Pattern formation, cardiac dynamics, and spiral waves
Phys 7682 / CIS 6229: Computational Methods for Nonlinear Systems

- patterns are ubiquitous in spatiotemporal systems driven out of equilibrium
  - regular, periodic patterns
  - localized, coherent structures ("defects")
The universality of patterns*

FitzHugh-Nagumo model

Rayleigh-Benard convection (Bodenschatz)

cAMP spiral waves in Dictyostelium chemotaxis

*see, e.g., Cross & Hohenberg, Rev. Mod. Phys. 65(3), 1993
Cardiac dynamics

normal electrocardiogram (ECG)

ECG during ventricular fibrillation

spiral waves implicated in arrhythmia:
incoherent pumping due to many different local pulse sources
FitzHugh-Nagumo model

- simple model of transmembrane voltages and currents in biological tissue
- reduction of Hodgkin-Huxley model of nerve conduction to two state variables

\[
\begin{align*}
\frac{\partial V}{\partial t} &= \nabla^2 V + \frac{1}{\epsilon} (V - V^3/3 - W) \\
\frac{\partial W}{\partial t} &= \epsilon (V - \gamma W + \beta),
\end{align*}
\]

\( V = \) transmembrane voltage

\( W = \) recovery variable

Cardiac tissue is an excitable medium

reaction-diffusion equation

nonlinear response within cells

diffusive coupling across cells
Spatially-independent FitzHugh-Nagumo model

\[
\begin{align*}
\frac{dV}{dt} &= V^2V + \frac{1}{\epsilon}(V - V^3/3 - W) \\
\frac{dW}{dt} &= \epsilon(V - \gamma W + \beta),
\end{align*}
\]

Nullcline analysis

V-nullcline: curve on which \(dV/dt = 0\)
W-nullcline: curve on which \(dW/dt = 0\)

use root-finder to find point \((V^*, W^*)\)
at which \((dV/dt, dW/dt) = 0\)

voltage pulses followed by refractory period
Two-dimensional FitzHugh-Nagumo model

\[
\frac{\partial V}{\partial t} = \nabla^2 V + \frac{1}{\epsilon} \left( V - \frac{V^3}{3} - W \right)
\]

\[
\frac{\partial W}{\partial t} = \epsilon \left( V - \gamma W + \beta \right)
\]

Finite-difference approximations

\[
\frac{\partial^2 V}{\partial x^2} \approx \frac{V(x + dx, y) - 2V(x, y) + V(x - dx, y)}{dx^2}
\]

\[
\frac{\partial^2 V}{\partial y^2} \approx \frac{V(x, y + dy) - 2V(x, y) + V(x, y - dy)}{dy^2}
\]

\[
\nabla^2 V_{i,j} \approx \frac{V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j}}{dx^2}
\]

\[
\nabla^2 V_{i,j} \approx \frac{1}{dx^2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]
\[ \nabla^2 V_{i,j} \approx \frac{1}{dx^2} \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{array} \right) \]

instead of:

```python
for i in range(1,Nx-1):
    for j in range(1,Ny-1):
```

use array operations to overlay shifted copies of array

\[ \nabla^2 V_{i,j} \approx \frac{1}{dx^2} \left( \begin{array}{ccc} 1/6 & 2/3 & 1/6 \\ 2/3 & -10/3 & 2/3 \\ 1/6 & 2/3 & 1/6 \end{array} \right) \]

stencils not unique
(same order, better fidelity)
Extensions of the basic model

Niels F. Otani, Further exploration of the FitzHugh-Nagumo model (Project Topics, Section 5, p. 24)

\[
\begin{align*}
\frac{\partial V}{\partial t} &= \nabla^2 V + \frac{1}{\epsilon} (V - V^3/3 - W) \\
\frac{\partial W}{\partial t} &= \epsilon (V - \gamma W + \beta)
\end{align*}
\]

spatially-varying parameters
(inc. diffusive coupling D)

Dead tissue

Source-sink characteristics

Two-chamber geometry with pacemaker
(sino-atrial node)