1. Schröder 3.32

(a) In moving the piston through a distance of \( \Delta x = 1 \text{ mm} \), the work done on the system is \( W = F \Delta x = 2000 \times 10^{-3} = 2 \text{ J} \).

(b) We recall that a fast compression is an adiabatic process. Therefore, no heat is added to the gas.

(c) Assuming that all the work done on the system is converted into the internal energy of the gas, the increase of internal energy is therefore \( \Delta U = 2 \text{ J} \).

(d) Calculating the change in entropy appears to be complicated because all of our familiar thermodynamic variables change in the process. That is, \( P_i, V_i, U_i, T_i \) are all different from \( P_f, V_f, U_f, T_f \). As is always the case, there are several ways to calculate the change in entropy. Schröder suggests that you use the thermodynamic identity \( dU = T \, dS - P \, dV \) to calculate the change in entropy. In this case, we want to look at the process in the \( U-V \) plane as outlined on pp. 111-112 of Schröder. Our process is one in which \( U_i < U_f \) and \( V_i > V_f \), so the two can be represented by two points in a figure like Fig. (3.15), except that – contrary to the figure – \( V_f \) is below \( V_i \) in our case. We envision replacing the actual process by two quasistatic processes, in which Step 1 is an increase in \( U \) from \( U_i \) to \( U_f \) with \( V \) held constant (as in Fig. 3.15) and Step 2 is a compression or step downward from \( V_i \) to \( V_f \) with \( U \) held constant at \( U_i \) (just opposite to Step 2 in Fig. 3.15). Since \( V \) is constant in Step 1, we have \( dU = T \, dS \) and

\[
\Delta S_1 = \int \frac{dU}{T} = C_V \int \frac{dT}{T} = C_V \ln \left( \frac{T_f}{T_i} \right).
\]

Now, \( T_f = T_i + \Delta U/C_V \), so \( \ln(T_f/T_i) = \ln[1 + \Delta U/(C_V T_i)] \approx \Delta U/(C_V T_i) \) and \( \Delta S_1 \approx \Delta U/T_i = W/T_i \). During Step 2, we have \( T \, dS = P \, dV \) since \( U \) (ergo \( T \)) is constant. Hence, we have

\[
\Delta S_2 = \frac{NkT}{T} \int \frac{dV}{V} = Nk \ln \left( \frac{V_f}{V_i} \right) = Nk \ln \left[ 1 + \frac{\Delta V}{V_i} \right] \approx Nk \frac{\Delta V}{V_i}.
\]

(Note \( \Delta V < 0 \)). Since \( Nk/V_i = P_i/T_i \), we have \( \Delta S_2 \approx P_i \Delta V/T_i \). Putting these together we find that the total change in entropy \( \Delta S \) is, \( \Delta S = \Delta S_1 + \Delta S_2 \approx (W + P_i \Delta V)/T_i \). Then, \( \Delta V = -A \Delta x = -10^{-2} \text{ m}^2 \times 10^{-3} \text{ m} = -10^{-5} \text{ m}^3 \), so \( P_i \Delta V \approx -1 \text{ J} \). Finally, \( \Delta S \approx (2 \text{ J} - 1 \text{ J})/(300 \text{ K}) = 3.33 \times 10^{-3} \text{ J/K} \). This is positive as it should be since the process that is got us from \( i \) to \( f \) is sudden – anything but quasistatic. It is possible to calculate \( \Delta S \) in the more familiar \( P-V \) plane. In this case you also need to define a third point in order to get from \( i \) to \( f \) along a quasistatic path. You could use a quasistatic isothermal or a quasistatic adiabatic process along with a quasistatic process at constant volume. You could also use both an isothermal and an adiabatic process. Since any of these choices require calculating \( P \) and \( V \) at this third point, they turn out to be more complicated analytically than the calculation in the \( U-V \) plane where quasistatic paths with one variable held constant are simple.

2. Schröder 4.1

(a) For an ideal diatomic gas, the number of degrees of freedom is \( f = 5 \). Thus, in the expression for the heat capacity at constant volume, \( c_v = \frac{1}{2} f Nk \), \( f \) is 5. With this, we can write the infinitesimal change in the internal energy as \( dU = c_v \, dT \). Referring to figure (1), absorption of heat occurs only along paths 1-2 and 2-3. Along 1-2, since this is an isochoric process, then heat absorbed is simply
given by $Q_{12} = \int_{1}^{2} c_v \, dT = \frac{f}{2} N k (T_2 - T_1) = \frac{f}{2} P_1 V_1$. Along the isobaric curve 2-3, one has $Q_{23} = \int_{2}^{3} (c_v \, dT + P_2 \, dV) = \frac{f}{2} N k (T_3 - T_2) + P_2 (V_3 - V_2) = (2f + 4) P_1 V_1$. The total heat absorbed is thus $Q = Q_{12} + Q_{23} = (5f + 8) P_1 V_1 / 2$. The total work done by the gas in one cycle is the area enclosed by the curve on the $P$-$V$ diagram. This is easily found to be $W = (P_2 - P_1) \times (V_2 - V_1) = 2 P_1 V_1$. The efficiency of the engine is therefore $\epsilon = W / Q = 4 / (5f + 8) = 0.121$.

(b) For an ideal engine operating between the same temperature extremes, the temperature at the hotter reservoir can be found from the expression $P_3 V_3 = N k T_h$. Similarly, the temperature at the colder extreme can be found from $P_1 V_1 = N k T_c$. The temperature ratio is then $T_c / T_h = P_1 V_1 / (P_3 V_3) = 1 / 6$. The efficiency of the ideal engine is then $\epsilon = 1 - T_c / T_h = 5 / 6 \approx 0.833$.

3. Schroeder 4.3

(a) The rate of useful work done by the power plant is $W = 1$ GW. From $\epsilon = W / Q_h$, the amount of heat needed per unit time to achieve this amount of work is then $Q_h = W / \epsilon = 2.5$ GW. The amount of heat released per unit time is therefore $Q_c = Q_h - W = 1.5$ GW.

(b) At $25^\circ$C, the specific heat of water is $c_w = 4.2 \times 10^3$ Jkg$^{-1}$K$^{-1}$. A flow rate of 100 m$^3$s$^{-1}$ corresponds to a mass rate of $m = 10^5$ kgs$^{-1}$. The amount of heat that the water has to absorb is, from previous,
\( Q_c = 1.5 \times 10^8 \text{ W}. \) Assuming that this amount of heat is entirely absorbed by water, we must have \( Q_c = m c_w \Delta T. \) From this, we get \( \Delta T = Q_c/(m c_w) = 3.57 \text{ K}. \)

(c) At 25\(^\circ\)C, the latent of evaporation of water is \( L = 580 \text{ cal/g} = 2.436 \times 10^6 \text{ J/kg}. \) Again, assuming that water is solely responsible for the absorption of heat, we then have \( Q_c = m L. \) The rate of evaporation is thus \( m = Q_c/L = 615.8 \text{ kg/s}, \) or 0.6158 \( \text{ m}^3\text{s}^{-1}. \) The fraction of the river which must be evaporated is \( 0.6158/100 = 6.158 \times 10^{-3}. \)

4. Schroeder 4.4

(a) The maximum possible efficiency of an engine operating between these two temperatures is one which runs on Carnot cycles. The efficiency of this Carnot engine is \( \epsilon = 1 - T_c/T_h = 1 - (4 + 273)/(22 + 273) = 0.061. \)

(b) The amount of useful work done by the Carnot engine is the amount of electrical power generated, \( W = 1 \text{ GW}. \) The amount of heat absorbed is then given by \( Q_h = m c_w \Delta T = W/\epsilon = 16.39 \text{ GW}. \) The rate of process is therefore \( m = Q_h/(c_w \Delta T) = 1.639 \times 10^9/[4.2 \times 10^3 \times (22 - 4)] = 2.168 \times 10^5 \text{ kgs}^{-1}, \) or \( 2.168 \times 10^2 \text{ m}^3\text{s}^{-1}. \)

5. Schroeder 4.7

An air conditioner functions by removing heat from our room and some amount of work is done in disposing it elsewhere. We know that heat flows from higher temperature to lower temperature. In flowing from lower temperature to higher temperature, some amount of work has to be done on the air conditioner. Together with this amount of work, a greater amount of wasted heat is disposed back into the room. So, if we put the air conditioner in the middle of the room, eventually the room will be warmer than we wanted.

6. Schroeder 4.8

Similar to the previous problem, we cannot cool down the same place where we dispose our wasted heat. By leaving the refrigerator door open, we are trying to cool down the air surrounding the refrigerator. But at the same time, the heat we are trying to remove from the kitchen will be disposed back to the same place again, in fact by a greater amount because some amount of work has to be done to remove heat from lower to higher temperature. Thus, the kitchen would be warmer than it previously was.

7. Schroeder 4.10

![Energy flow diagram for the refrigerator.]

For a typical kitchen refrigerator, \( T_h = 298 \text{ K} \) while \( T_c = 255 \text{ K}. \) The coefficient of performance for an ideal operation is \( \eta = T_c/(T_h - T_c) = 255/(298 - 255) = 5.93. \) The rate of heat leaking into the fridge is \( Q_c = 300 \text{ W}. \) From the definition of coefficient of performance, \( \eta = Q_c/W. \) Hence, the rate of power it must draw from the wall is \( W = Q_c/\eta = 300/5.93 = 50.59 \text{ W}. \)
8. (a) The efficiency of a Carnot engine is \( \epsilon = 1 - T_c/T_h = 1 - 300/400 = 1/4 = 0.25 \).

(b) From \( \epsilon = W/Q_h \), one finds \( W = \epsilon Q_h = 0.25 \times 1600 = 400 \) J.

(c) The total heat absorbed along the isotherm with \( T_h = 400 \) K can be calculated by applying the first law \( dQ = dU + PDV \). Since along the isotherm \( dU = 0 \), the total heat absorbed is simply
\[
Q_{12} = \int_{1}^{2} P dV = N k T_h \ln(V_2/V_1).
\]
Equating this expression with \( Q_h = 1600 \) J and using \( V_2/V_1 = 4 \), one finds \( N k = Q_{12}/[T_h \ln(V_2/V_1)] = 1600/(400 \ln 4) = 2.885 \) J K\(^{-1}\).

(d) Along the isotherm 1-2, we can apply the ideal gas law and find \( P_1/P_2 = V_2/V_1 = 4 \), or \( P_2 = P_1/4 \).

(e) For isothermal paths going from point \( i \) to \( j \), the work done by the gas is \( -W_{ij} = N k T_i \ln(V_j/V_i) \). Whereas for adiabatic paths going from \( i \) to \( j \), the work done by the gas is \( -W_{ij} = N k (T_i - T_j)/(\gamma - 1) \). We also note that, by the first law, the heat added to the gas in an isothermal process is \( Q_{ij} = -W_{ij} \), and from the definition of adiabatic process \( Q_{ij} \) along any adiabatic path \( i-j \) must be zero. The entropy change along any path \( i-j \) can be calculated from \( S_{ij} = \int_{i}^{j} dQ_{ij}/T \), where \( Q_{ij} \) is evaluated along a reversible path, in our case it is either an adiabat or isotherm. Applying the first law and then using \( dU = c_v dT \) and the ideal gas law, one finds
\[
S_{ij} = N k \left[ \frac{f}{2} \ln\left( \frac{T_j}{T_i} \right) + \ln\left( \frac{V_j}{V_i} \right) \right].
\]

The values for \( -W_{ij}, Q_{ij} \) and \( S_{ij} \) for each leg of the Carnot cycle is tabulated in table (2)

<table>
<thead>
<tr>
<th>Paths ( i-j )</th>
<th>(-W_{ij})</th>
<th>(Q_{ij})</th>
<th>(S_{ij})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>1600 J</td>
<td>1600 J</td>
<td>4 JK(^{-1})</td>
</tr>
<tr>
<td>2-3</td>
<td>432.75 J</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3-4</td>
<td>-1200 J</td>
<td>-1200 J</td>
<td>-4 JK(^{-1})</td>
</tr>
<tr>
<td>4-1</td>
<td>-432.75 J</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: \(-W_{ij}, Q_{ij} \) and \( S_{ij} \) for each leg \( i-j \) of the Carnot cycle.

(f) On a \( T-S \) diagram with \( T \) being the vertical axis and \( S \) being the horizontal axis, isotherms are straight horizontal lines and adiabats are straight vertical lines. This is shown in figure (4). The difference between the maximum and minimum \( S \) is \( \Delta S = S_{\text{max}} - S_{\text{min}} = 4 \) JK\(^{-1}\).
(g) The area enclosed by the Carnot cycle in the $P$-$V$ diagram is $A_{PV} = \sum -W_{ij} = 1600 - 1200 = 400$ J. Equating this to $A_{TS} = (T_h - T_c) \Delta S = 100 \Delta S$, one finds $\Delta S = 4$ JK$^{-1}$.

**Rigmarole:** Beginning students of thermodynamics usually find that the mathematics behind this subject to be cumbersome, if not frustrating. A large part of the time is spent in manipulating partial derivatives and differentials in a form that few could remember any length of time after the subject was taught. We may note as an example that in mechanics and electrodynamics, the use of coordinate systems usually makes our equations of motion very cumbersome (just recall expressing $\nabla^2$ in spherical coordinate system!). This agony has partly been removed by the introduction of tensor notation and differential forms. Similar effect can be achieved by the use of Jacobian determinant in thermodynamics. It enables one to express physical relations without referring to any particular set of thermodynamical variables. This is perhaps one of the few cases of a retrograde progress in physics. History has it that the use of Jacobian determinant was the original mathematical formulation of thermodynamics. One of the founders of thermodynamics, Clausius, made extensive use of this notion in his works, though expressed in a controversial notation. It was probably due to this bad choice of notation that the Jacobian determinant went out of favor. The modern notation we adopt today has largely made it accessible again to a larger audience. Here is a flavour of how it works. We define the Jacobian determinant of the transformation from $P$-$Q$ to $x$-$y$ as

\[
[P, Q] = \frac{\partial(P, Q)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{vmatrix}.
\]

The Jacobian matrix, which forms this determinant, is sometimes simply called the Jacobian. The determinant is, confusingly and unfortunately, often called the Jacobian as well. This confusion should not bother us here. The use of this notation immediately allows one to carry out derivations without referring to any particular coordinate systems, since in changing to another set of variables
$x'-y'$, the Jacobian determinant has the property that
\[
\frac{\partial(P,Q)}{\partial(x',y')} = \frac{\partial(P,Q)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(x',y')} \,.
\]  
(1)

or, in the more compact notation, is simply $[P,Q]' = [P,Q][x,y]'$. In geometry, this is related to the conservation of volume element, the volume of a sphere has to be the same regardless of whether you evaluate it in a Cartesian or spherical coordinate system. One special case of equation (1) is when $Q = y' = \text{constant}$. Then, together with the property
\[
\frac{\partial(P,Q)}{\partial(x,y)} = \left[ \frac{\partial(x,y)}{\partial(P,Q)} \right]^{-1},
\]
one has
\[
\frac{\partial P}{\partial x'} = \frac{\partial(P,Q)}{\partial(x',Q)} = \frac{[P,Q]}{[x',Q]}.
\]  
(2)

Now, let us consider the four Maxwell relations
\[
\left( \frac{\partial T}{\partial V} \right)_S = -\left( \frac{\partial P}{\partial S} \right)_V, \quad \left( \frac{\partial T}{\partial P} \right)_S = \left( \frac{\partial V}{\partial S} \right)_P, \quad \left( \frac{\partial P}{\partial T} \right)_V = \left( \frac{\partial S}{\partial V} \right)_T, \quad \left( \frac{\partial V}{\partial T} \right)_P = -\left( \frac{\partial S}{\partial P} \right)_T.
\]

The advantage of Jacobian determinant is apparent when we apply equation (2) and the antisymmetric property $[P,Q] = -[Q,P]$ to the four Maxwell relations. We find that all of them reduce to a single equation
\[
[P,V] = [T,S] \quad \text{or} \quad \frac{\partial(P,V)}{\partial(T,S)} = 1.
\]

The last equation is just the statement that the transformation from the $P$-$V$ plane to the $T$-$S$ plane preserves the area. In particular, for a reversible cycle, in a more modern vector calculus notation, the preservation of area can be written as
\[
\oint P \, dV = \oint T \, dS,
\]
where both $\oint$ means integration along the closed loop of the cycle and the direction of integration must be consistent in both planes. Now, we learned that the area enclosed by a cycle in a $P$-$V$ diagram is just the total work done by the gas in one cycle. The closed-loop integral in $T$-$S$ plane is not so familiar to us, but we can tell, from the Carnot engine we worked out in this problem, that integration from lower entropy to higher entropy at a higher temperature gives us the heat absorbed by the gas from the hot reservoir $Q_h$, and integration from higher entropy to lower entropy at a lower temperature gives us the heat released by the gas to the cold reservoir $Q_c$, so the closed-loop integral is just $\oint T \, dS = Q_h - Q_c$. So the preservation of area implies that $W = Q_h - Q_c$, and this is the conservation of energy which is old and familiar to us! On a more unrelated tone, the notion that if a quantity (in our case here is the area) does not change under a transformation (be it coordinate transformations or some fancier transformations) then there must be a conservation law lurking (here it is the law of conservation of energy) is a notion that has many applications in many areas of physics.