Homework 9 solutions

1. Let’s first calculate the energy in 1 kg of matter. We have:

\[ E = mc^2 \]
\[ = (1 \text{ kg}) \cdot (3 \times 10^8 \text{ m/s})^2 \]
\[ = 9 \times 10^{16} \text{ J}. \]

Since one gallon of gasoline delivers roughly \( 1 \times 10^8 \) J, the volume, \( V \), of gasoline needed to yield this amount of energy is

\[ V = 9 \times 10^{16} \text{ J} \cdot \frac{1 \text{ gal}}{1 \times 10^8 \text{ J}} \]
\[ = 9 \times 10^8 \text{ gal}. \]

2. A clock passes you at \( v = .98c \).

(a) Let’s find \( \gamma \). We have \( \beta = v/c = .98 \).

\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \]
\[ = \frac{1}{\sqrt{1 - (.98)^2}} \]
\[ = \frac{1}{\sqrt{.0396}} \]
\[ = \frac{1}{0.199} \]
\[ = 5.03 \]

(b) The second hand of the clock takes 1 minute to make a complete revolution, as measured in the rocket. This is the proper time, \( T_0 \) (that is, it is the time measured by someone at rest with respect to the clock). We search for the time it takes for an observer on Earth. Time is dilated:

\[ T = \gamma T_0 \]
\[ = 5.03 \cdot 1 \text{ min} \]
\[ = 5.03 \text{ min} \]
(c) A meter stick, as measured by someone in the rocket is, well, 1 meter long! This is the proper length, $L_0$. We want to find the length as measured on Earth. We only have to use the length contraction formula:

$$L = \frac{L_0}{\gamma}$$

$$= \frac{1 \text{ m}}{5.03}$$

$$= .199 \text{ m}$$

3. The star $\alpha$-Centuri is 4 L.Y. from the Earth.

(a) By definition of “light years”, it takes 4 years for light to travel the distance between Earth and $\alpha$-Centuri.

(b) Since we are at rest with respect to the Earth and $\alpha$-Centuri, the distance we measure between them is actually the proper distance. So, $L_0 = 4$ L.Y. Now, as seen from the rocket, the Earth and the star are moving at $v = .98c$. Therefore, the distance as measured in the rocket is shorter (this is length contraction):

$$L = \frac{L_0}{\gamma}$$

$$= \frac{4 \text{ L.Y.}}{5.03}$$

$$\approx .8 \text{ L.Y.}$$

(c) As seen from the rocket, $\alpha$-Centuri is traveling towards it at $v = .98c$. The time the star needs to reach the rocket is therefore the time it takes to travel a distance of .8 L.Y. at this speed:

$$t = \frac{d}{v}$$

$$= \frac{.8 \text{ L.Y.}}{.98 \cdot 1 \text{ L.Y./Yr}}$$

$$\approx 0.81 \text{ Yr},$$

where we have used the fact that the speed of light is one light-year per year ($c = 1 \text{ L.Y./Yr}$). Note that there is another way to find this time. As measured from Earth, the time to reach $\alpha$-Centuri is $d/v = 4\text{L.Y./}.98c = 4.08 \text{ Yr}$. But since we know that time in the rocket is “flowing” slower, a time interval of 4.08 Yr on Earth corresponds to only $4.08 \text{ Yr}/5.03 = 0.81 \text{ Yr}$ in the rocket.

(d) Of course, nothing but light (or other massless particles) can reach the speed of light. Also, if we set $v = c$, we get $\gamma = 1/0$, which is not definite. The way you can handle this problem is to think of the limit when $v$ goes to $c$. Let’s look at the second way we solved the preceding problem. We
first found that the time it would take for an object going at \( v = .98c \) to reach \( \alpha\)-Centuri is 4.08 years. If \( v \) increases, we can easily see that the travel time to \( \alpha\)-Centuri is getting closer and closer to 4 years. Thus in the limit \( v \to c \), the travel time as measured on Earth is 4 years. Now, what is the travel time as measured in the rocket? You know that \( \gamma \) increases very fast when \( v \) gets closer and closer to \( c \). This means that the travel time, as measured in the rocket, is getting smaller and smaller. In the limit, \( \gamma \) is infinite and the travel time for the rocket is zero!

If you want, you can also see it as a length contraction problem: for the observer in the rocket, the distance she has to travel is getting shorter and shorter (since \( L = L_0 / \gamma \)). In the limit \( v \to c \), all the universe is squeezed in an infinitely small distance, and it takes only a minute amount of time to travel through it.

4. Problem 1, p. 320. Mort measures the length of Velma’s spaceship as 1/2 its proper length. This means that \( L / L_0 = 1/2 = 1/\gamma \), or \( \gamma = 2 \). Let’s find the speed of the spaceship:

\[
\gamma = \frac{1}{\sqrt{1 - \beta^2}} = 2
\]

\[
\frac{1}{2} = \sqrt{1 - \beta^2}
\]

\[
\frac{1}{4} = 1 - \beta^2
\]

\[
\beta^2 = 1 - \frac{1}{4}
\]

\[
\beta = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \approx 0.866
\]

Thus, Velma’s spaceship is going at a speed \( v = .866c \) relative to Mort.

Since \( \gamma \) is only a function of relative velocity, \( \gamma = 2 \) for Velma as well. According to her, the United States are 2500 km wide (5000 km / \( \gamma \)).

5. Problem 19, page 320. The rest mass of a the meter stick is 1 kg. It is measured to have a 2 kg mass (laboratory mass). The equation that relates the laboratory mass and the rest mass is \( m = \gamma m_0 \). Thus, \( \gamma = m / m_0 = 2 \). Now that you know \( \gamma \), you can find the speed at which the meter stick is passing by you. But here, it is not even necessary: we are searching for its length as measured by you. This length is contracted by an amount \( 1/\gamma \). Thus, you measure the length of the meter stick as 50 cm.
6. The equation of relativistic addition of speeds is

\[ v = \frac{v' + u}{1 + \frac{uv'}{c^2}} \]

The only difficulty in this problem is about how you define \( v \), \( u \) and \( v' \). First, let’s define \( u \). This is the relative speed of the two reference frames. In this case, these two frames are the rocket and the platform and their relative speed is \(.9c\). \( v \) is the velocity measured from the platform’s reference frame, while \( v' \) is the velocity measured from the rocket. You know that \( v' = .7c \), therefore you can find \( v \):

\[ v = \frac{.7c + .9c}{1 + \frac{(7c)(9c)}{c^2}} = \frac{1.6c}{1 + .63} \approx 1.6c \approx 0.982c \]

As expected, the velocity is still smaller than \( c \).