I. STERN-GERLACH EXPERIMENT

A spin $\frac{1}{2}$ particle approaches a Stern-Gerlach (S-G) apparatus from the left and enters it at $t = 0$. It is in the spin-up state.

1. Write down its spinor wavefunction at the point of entry.

2. The Hamiltonian for the spin $\frac{1}{2}$ electron is

$$H = \begin{cases} 
0 & t < 0 \\
-\mu \cdot B = -\gamma (B_0 + \alpha z) S_z & 0 < t < T \\
0 & t > T 
\end{cases}$$

where $T$ is the time for it to pass through the S-G apparatus, $B_0$ is the magnitude of the $B$-field along the $z$-axis, and $\alpha$ is the magnitude of the $B$-field gradient. Write down the complete spinor wavefunction for the electron just at the point of exit of the apparatus.

3. Considering only components in this wave function of the form $e^{ikz}$, write down the momentum along the $z$-direction which the electron picks up by virtue of passage through the S-G apparatus.

4. What momentum would the electron have picked up if it were spin down upon entry?

II. INFINITE SPHERICAL WELL

The radial component of the Schrödinger equation in spherical coordinates is

$$\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \frac{2\mu r^2}{\hbar^2} \left[ V(r) - E \right] R = \ell(\ell + 1)R,$$

where $\Psi = RY_m^\ell$.

1. Change variables to $R = \frac{u(r)}{r}$ and show that the Radial equation can now be written as

$$-\frac{\hbar^2}{2\mu} \frac{d^2 u}{dr^2} + \left[ V(r) + \frac{\hbar^2 \ell(\ell + 1)}{2\mu r^2} \right] u = Eu.$$

2. Consider the infinite spherical well potential

$$V(r) = \begin{cases} 
0 & r < a \\
\infty & r > a 
\end{cases}.$$
and show that inside this well the radial equation can be written as

\[
\frac{d^2 u}{dr^2} = \left[ \frac{\ell(\ell + 1)}{r^2} - k^2 \right] u.
\]

3. Find an equation for \( k^2 \) in terms of \( E, \mu \) and \( \hbar^2 \).

4. Show that the \( \ell = 0 \) wavefunction is \( u = A \sin(kr) \).

5. The boundary condition means that \( u = 0 \) for \( r > a \); find the energy levels for \( \ell = 0 \) states.

III. ANGULAR MOMENTUM AND SPHERICAL HARMONICS

Two particles of mass \( m \) are attached to the ends of a massless rigid rod of length \( a \). The system is free to rotate about the center. [Don’t worry about whether the particles are identical or not: just assume you can tell them apart. If you’re bored, think about whether it would make a difference.]

1. Show that the allowed energies of this rigid rotator are

\[
E_n = \frac{\hbar^2 n(n + 1)}{ma^2},
\]

for integers \( n \). [Hint: Express the classical energy in terms of angular momentum.]

2. Write down the wavefunctions for \( n = 2 \).