1. Reading Assignment

a) This problem set covers Serway 26, 28.4-28.6
b) For next week, we will cover material in chapter 29

2. Serway 26-55A

A parallel-plate capacitor in air has a plate separation $d$ and a plate area $A$. The plates are charged to a potential difference $V$ and disconnected from the source. The capacitor is then immersed in a liquid of dielectric constant $\kappa$. Determine

a) the charge on the plates before and after immersion,

b) the capacitance and voltage after immersion and

c) the change in energy of the capacitor.

SOLUTION:

a) Before immersion: $Q = CV = \varepsilon_0 (A/d)V$. The charge remains the same after immersion.

b) After immersion the capacitance is $\kappa \varepsilon_0 A/d$, and the voltage is $V_f = V/\kappa$, since the product of capacitance and voltage still has to be $Q$.

c) The energy stored in a capacitor is $U = (1/2)CV^2 = (1/2)QV$. The change in stored energy is

$$U_f - U_i = \frac{1}{2} QV_f - V_i = \frac{1}{2} \varepsilon_0 \frac{AV}{d} \left( \frac{V_f}{\kappa} - V \right) = \varepsilon_0 \frac{AV^2}{2d} \left( \frac{1}{\kappa} - 1 \right)$$

After immersion the stored energy is less than before.

Comment: If $V$ had been kept fixed rather than $Q$, the stored energy would have increased upon immersion into the dielectric.

3. Serway 26-72

A capacitor is constructed from two square plates of sides $l$ and separation $d$. A material of dielectric constant $\kappa$ is inserted a distance $x$ into the capacitor, as in Figure P26.72 of your text.

a) Find the equivalent capacitance of the device.

b) Calculate the energy stored in the capacitor if the potential difference is $V$.

c) Find the direction and magnitude of the force exerted on the dielectric, assuming a constant potential difference $V$. Neglect friction and edge effects.

d) Obtain a numerical value of the force assuming that $l = 5.0$ cm, $V = 2000$ V, $d = 2.0$ mm, and the dielectric is glass ($\kappa = 4.5$).

(Hint: The system can be considered as two capacitors connected in parallel.)

SOLUTION:

a) Add the two parallel capacitances. Their areas are $lx$ and $l(l-x)$, respectively.

$$C = \kappa \varepsilon_0 \frac{lx}{d} + \varepsilon_0 \frac{l(l-x)}{d}$$

b) $U = \frac{1}{2} CV^2 = \varepsilon_0 \frac{lV^2}{2d} (\kappa - 1) x$.

c) The potential energy is a function of $x$, the position of the dielectric. The force on the dielectric is equal to the negative gradient of the potential energy

$$\vec{F} = -\frac{\partial U}{\partial x} \hat{x} = -\frac{\varepsilon_0 lV^2}{2d} (\kappa - 1) \hat{x}$$
Since $\kappa > 1$, the force points to the left, in a direction to push the dielectric out of the capacitor. Qualitatively we can predict this by noting that the capacitance, and for fixed $V$, the potential energy is less for smaller $x$, that is if the dielectric is inserted less far.

d) 
\[
F = \frac{8.85 \times 10^{-12} \text{C}^2/\text{Nm}^2 \times 0.05 \text{ m} \times (2000 \text{ J/C})^2 \times (4.5 - 1)}{2 \times 0.002 \text{ m}} = 1.55 \times 10^{-3} \text{ N}
\]
A parallel-plate capacitor with plates of area $A$ and plate separation $d$ has the region between the plates filled with two dielectric materials as shown in Figure P26.84 in your text.

a) Determine the capacitance, and

b) show that when $\kappa_1 = \kappa_2 = \kappa$, your result becomes the same as that for a capacitor containing a single dielectric; $C = \kappa \varepsilon_0 A/d$.

SOLUTION: Since there are a lot of differentials in this problem, it is convenient to change the name of the gap width from $d$ to $a$.

a) Let us put the capacitor parallel to the $x$–axis with its lower left corner at $x = 0, y = 0$. In our figure, the slope of the line separating the two dielectrics is $a/L$. A small slice of the capacitor of area $Wdx$ at a location $x$ consists of two capacitors in series, $dC_2$ with dielectric $\kappa_2$ and average gap width $y = ax/L$ and $dC_1$ with dielectric $\kappa_1$ and average gap width $a - y = a - ax/L = a(L - x)/L$. Neglecting that the border between the two is at an angle, we have

$$dC_2 = \varepsilon_0 \kappa_2 \frac{Wdx}{y} = \varepsilon_0 \kappa_2 \frac{LWdx}{ax}$$

$$dC_1 = \varepsilon_0 \kappa_1 \frac{Wdx}{a - y} = \varepsilon_0 \kappa_1 \frac{LWdx}{a(L - x)}$$

and the series capacitance $dC$ of this slice is given by

$$dC = \left( \frac{1}{dC_2} + \frac{1}{dC_1} \right)^{-1}$$

$$= \frac{\varepsilon_0 WL}{a \kappa_2 L + (\kappa_1 - \kappa_2)x}$$

To find the total capacitance $C$ of the assembly we have to integrate

$$C = \int dC = \left[ \varepsilon_0 \frac{WL}{a \kappa_2 L + (\kappa_1 - \kappa_2)x} \right]_0^{\kappa_1 L}$$

Substituting $u = \kappa_2 L + (\kappa_1 - \kappa_2)x$, $du = (\kappa_1 - \kappa_2)dx$

$$C = \varepsilon_0 \frac{WL}{a} \int_{\kappa_2 L}^{\kappa_1 L} \frac{du}{\kappa_2 L + (\kappa_1 - \kappa_2)u} = \varepsilon_0 \frac{WL}{a} \frac{\kappa_1 \kappa_2}{\kappa_1 - \kappa_2} \ln \frac{\kappa_1}{\kappa_2}$$

(4.1)

b) As $\kappa_1 \to \kappa_2$ (4.1) becomes

$$C = \varepsilon_0 \frac{WL}{a} \lim_{\kappa_1 \to \kappa_2} \frac{\ln \kappa_1 - \ln \kappa_2}{\kappa_1 - \kappa_2}$$

$$= \varepsilon_0 \frac{WL}{a} \kappa_2 \frac{d \ln \kappa}{d \kappa} = \varepsilon_0 \frac{WL}{a}$$

This is the capacitance of a parallel plate capacitor with dielectric constant $\kappa$, area $WL$, and gap width $a$.

5. Serway 28-42

Consider a series $RC$ circuit (Figure 28.15 in your text) for which $R = 1.00\,\text{MΩ}$, $C = 5.00\,\mu\text{F}$, and $E = 30.0\,\text{V}$. Find
a) the time constant of the circuit and

b) the maximum charge on the capacitor after the switch is closed.

c) If the switch is closed at $t = 0$, find the current in the resistor 10.0 s later.

**SOLUTION:**

a) The time constant is $\tau = RC = 1.00 \times 10^6 \Omega \times 5.00 \times 10^{-6} \text{ F} = 5.00 \text{ s}.

b) The full voltage $E$ is across the capacitor when its charge has reached its maximum value. $Q_{\text{max}} = CE = 5.00 \mu\text{F} \times 30.0 \text{ V} = 150 \mu\text{C}.$

c) At $t = 0$, the full voltage $E$ is across the resistor, since there is no charge yet on the capacitor. The initial current is $i_0 = E/R$. The current decreases exponentially with time constant $\tau = 5 \text{ s}$ as the capacitor is charged. At $t = 10 \text{ s}$

$$i = i_0 e^{-t/\tau} = i_0 e^{-2} = \frac{30 \text{ V}}{1.00 \ \text{M\Omega}} \times 0.1353 = 4.06 \mu\text{A}$$

6. Serway 28-51A

Dielectric materials used in the manufacture of capacitors are characterized by conductivities that are small but not zero. Therefore, a charged capacitor slowly loses its charge by “leaking” across the dielectric. If a capacitor having capacitance $C$ leaks charge such that the potential difference decreases to half its initial value in a time $t$, what is the equivalent resistance of the dielectric?

**SOLUTION:** The charge and the voltage decrease exponentially (see Figure 28.17(b) and Eq. (28.15) in your text).

$$\frac{Q}{Q_0} = \frac{V}{V_0} = 0.5 = e^{-t/RC}$$

$$\ln 0.5 = -\ln 2 = -t/RC$$

$$R = \frac{t}{C \ln 2}$$

7. Serway 28-72

Before the switch is closed in Figure P28.72 in your text, there is no charge stored by the capacitor. Determine the currents in $R_1$, $R_2$ and $C$

a) at the instant the switch is closed (that is, at $t = 0$) and

b) after the switch is closed for a long time (that is, as $t \to \infty$).

**SOLUTION:**

a) At $t = 0$ there is no charge yet on the capacitor, and therefore no voltage across $R_2$ which is connected in parallel. The full voltage is across $R_1$. At $t = 0$

$$i_1 = \frac{10 \text{ V}}{10 \ \text{k\Omega}} = 1.0 \text{ mA}$$

$$i_2 = 0$$

b) As $t \to \infty$ the capacitor is fully charged. Current flows through the series combination of $R_1$ and $R_2$ only and

$$i_1 = i_2 = \frac{10 \text{ V}}{10 \ \text{k\Omega} + 5 \ \text{k\Omega}} = 0.67 \text{ mA}$$

**NOTE:**

One can analyze this circuit more generally by applying Kirchoff’s Laws. Let $I_1$ be the current through $R_1$, let $I_c = dq/dt$ be the current charging the capacitor, and let $I_2$ be the current through $R_2$. Note that all of these currents will vary with time in this circuit.

Application of Kirchoff’s Laws results in the following three equations:
1. \( I_1 = I_c + I_2 = dq/dt + I_2 \) or \( I_2 = I_1 - dq/dt \)
2. \( V - I_1 R_1 - q/C = 0 \) (inner loop).
3. \( V - I_1 R_1 - I_2 R_2 = 0 \) (outer loop)

By combining (1) and (3), we find

\[
I_1 = \frac{V + R_2 \frac{dq}{dt}}{R_1 + R_2}
\]

and substitution of this expression into (2) yields

\[
V \left( 1 - \frac{R_1}{R_1 + R_2} \right) - \frac{R_1 R_2}{R_1 + R_2} \frac{dq}{dt} - q/C = 0
\]

Note that this is the same form as Equation 28.9 in Serway for a simple series circuit with a resistance \( R_c \), capacitance \( C \), and a voltage source \( V_c \), with

\[
V' = V \left( 1 - \frac{R_1}{R_1 + R_2} \right) = V \frac{R_2}{R_1 + R_2}
\]

and

\[
R_c = \frac{R_1 R_2}{R_1 + R_2}
\]

As in the development by Serway, we can rearrange the above to obtain

\[
\frac{dq}{CV' - q} = \frac{dt}{CR_c}
\]

integrating yields

\[- \ln(CV' - q) = \frac{t}{CR_c} + A\]

and finally, exponentiating and evaluating the constant of integration \( A \) by using the initial condition \( q(0) = 0 \) gives the final solution

\[
q(t) = CV' \left( 1 - e^{-\frac{t}{CR_c}} \right) = CV \frac{R_2}{R_1 + R_2} \left( 1 - e^{-\frac{t}{CR_1 R_2}} \right)
\]

We can obtain the current \( I_c = dq/dt \) from our solution:

\[
\frac{dq}{dt} = \frac{V}{R_1} e^{-\frac{t(R_1 + R_2)}{CR_1 R_2}}
\]

and substitution into our previous equations for \( I_1 \) and \( I_2 \) yields

\[
I_1 = \frac{V}{R_1} \frac{R_1 + R_2 e^{-\frac{t(R_1 + R_2)}{CR_1 R_2}}}{R_1 + R_2}
\]

\[
I_2 = \frac{V}{R_1 + R_2} \left( 1 - e^{-\frac{t(R_1 + R_2)}{CR_1 R_2}} \right)
\]

For the two limits \( t = 0 \) and \( t \to \infty \), the above formula yield the same results we obtained above.