Renormalization of the Ising model, part II

Due date: Thursday, November 1

1. The decimation procedure you used in the previous assignment is doomed to fail because it lacks an essential rescaling of the spins themselves! This omitted rescaling operation is called “wave function renormalization”. In honor of Leo Kadanoff’s 75th birthday, this first problem introduces a clever technique he devised, where spins are rescaled in the course of decimation.

Notation: Primed and unprimed quantities refer to two distinct square-lattice Ising models:

- unprimed model
  - \( r \) : lattice site
  - \( H\{s\} \): Hamiltonian \( \times \beta \)
  - \( s_r \): spin at \( r \)

- primed model
  - \( r' \) : lattice site
  - \( H'\{s'\} \): Hamiltonian \( \times \beta \)
  - \( s'_r \): spin at \( r' \)

The spins in the two models are paired-up so that the primed sites correspond to just one of the checkerboard colors of the unprimed sites. To keep the notation simple, when we write the product \( s_r s'_r \) we mean the product of spins from different models but whose positions coincide. The primed square lattice is rotated relative to the unprimed one by 45° and expanded in scale by \( b = \sqrt{2} \).

Kadanoff’s decimation scheme is defined by the following equation:

\[
\exp ( -H'\{s'\} ) = \sum_{\{s\}} \prod_{r'} \frac{1}{2} \left( 1 + \rho s_r s'_r \right) \exp ( -H\{s\} ).
\]

The real parameter \( \rho \) generalizes the standard decimation scheme, which corresponds to \( \rho = 1 \). For just that case the sum over all unprimed spin configurations omits all terms where the spins on one checkerboard color do not all agree with the paired spins of the primed model (since then at least of the \( (1 + s_r s'_r) \) factors would be zero). We will see later that the appropriate value of \( \rho \) is greater than 1, thereby introducing negative factors for all the unequal paired spins (since \( 1 - \rho < 0 \)). Thus there is something mysteriously fermonic behind Kadanoff’s idea!
(a) Show that for arbitrary $\rho$

$$\sum_{\{s'\}} \exp \left( -H'\{s'\} \right) = \sum_{\{s\}} \exp \left( -H\{s\} \right).$$

The mapping $R : H \rightarrow H'$ defined by this equation is the renormalization map.

(b) A very direct way to see the need for $\rho \neq 1$ is to consider the spin-spin correlation function:

$$G(r_1, r_2) = \langle s_1 s_2 \rangle_H.$$

Show that

$$G(r_1, r_2) = \frac{1}{\rho^2} G'(r'_1, r'_2).$$

This suggests defining a scaling dimension $y_s$ for spin by $\rho = b^{y_s}$.

(c) Suppose that for a particular $\rho^*$ we can choose parameters in $H$ so that the iterates $H$, $H'$, $H''$, etc. of $R$ flow toward a fixed-point $H^*$ — i.e. we start on the critical submanifold. By iterating the scaling transformation of the spin-spin correlation function show that

$$G(r_1, r_2) \sim \frac{G_0}{|r_1 - r_2|^{2y_s}}$$

for large $|r_1 - r_2|$. You may assume the correlation function becomes isotropic (depends only on $|r_1 - r_2|$) near the critical point.

(d) In this last part you will show that the fixed-point $H^*$ is not unique, that in fact there is a line of fixed Hamiltonians $H_\sigma$ under $R$, where $\sigma$ is a real parameter. Consider the family of Hamiltonians defined by

$$\exp \left( -H_\sigma\{\tilde{s}\} \right) = \sum_{\{s\}} \prod_{r} \frac{1}{2} \left( 1 + \sigma s_r \tilde{s}_r \right) \exp \left( -H^*\{s\} \right),$$

where $H^*$ is some fixed-point Hamiltonian (it doesn’t matter which). Note that spins $\{s\}$ and $\{\tilde{s}\}$ live on the same lattice this time. Show that, for any $\sigma$,

$$R(H_\sigma) = H_\sigma.$$
2. Consider the 1D Ising chain in a magnetic field:

\[-\beta H = K \sum_r s_r s_{r+1} + H \sum_r s_r.\]

(a) Using the transfer matrix, find the magnetization \(m\) per spin.
(b) Show that the singular part of the \(m\) satisfies

\[m_s = f(e^{-2K/H}),\]

and determine the scaling function \(f(x)\) explicitly.

3. Upon renormalization, the singular part of the magnetization of the \(D\)-dimensional Ising model satisfies

\[m_s(u_t, u_h) = b^{\mu_n - D} m_s(b^{\mu_n} u_t, b^{\mu_n} u_h).\]

(a) Show that for \(t \to 0, h \to 0^+\),

\[m_s(t, h) = h^{1/\delta} \Psi(t/h^{1/\Delta}),\]

where \(\Psi\) is a universal function analogous to the functions \(\phi_\pm\) and \(f\) discussed in lecture. Use the same conventions to fix the scale of \(\Psi\) and its argument:

\[m_s(t, 0) \sim |t|^\beta, \quad t \to 0^-\]

\[m_s(0, h) \sim h^{1/\delta}, \quad h \to 0^+ .\]

(b) Find an explicit functional relationship between \(\Psi\) and Widom’s ferromagnetic equation of state function \(f\):

\[\Psi^{-1}(z) = \cdots.\]

(c) Obtain the asymptotic behavior of \(\Psi(w)\) for \(w \to \pm \infty\). You may use part (b) and the behavior for \(f\) we found in lecture. Note: \(\Psi(w) \sim 0\) is not an asymptotic behavior; show how it vanishes.