PHYSICS 214
WAVES on a String; Sound, Light \( v_3 \)
OPTICS: Interference, Diffraction; Fibers, Films, & Slits \( v_6 \)
& PARTICLES: Quantum Mechanics \( \sqrt{2} \)

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PENDULUM DEMO: oscillate, spin, damping [ignored]
Hold at \( \theta_0 \sim 30^\circ \geq 0 \); drop. Catch after 2 periods.

Exercise 1: Draw \( \theta(t) \)

Exercise 2: Mark the period of the motion \( T \) on your graph. In terms of \( T \), what is the frequency \( F \) in cycles per second?

Exercise 3: During which time is the angular acceleration \( \frac{d^2\theta}{dt^2} < 0 \) when \( \theta < 0 \). Restoring force
The Equation of Motion for the Pendulum

Physics 112, Serway reading assignment

1. Free Body Diagram
2. Sum forces: set to 0 ma
3. Find differential equation satisfied by \( \theta(t) \)

**Equation of Motion**

\[
\frac{d^2\theta}{dt^2} = -\frac{g}{L} \sin \theta
\]

The equation of motion tells you how the velocity changes in the future, given the positions and velocities now.

We can't write down the exact solution to this differential equation. We need to find a numerical solution.

The solution is \( \theta(t) \) which solves the equation of motion.
We can't store $\theta(t)$ on the computer:

$$[8 \text{ bytes per } \theta] \times [\infty \text{ different times}] = \infty \text{ Gbyte hard disk}$$

We approximate $\theta(t)$ by a set of equally spaced points $\theta(t_n)$ where $t_n = n \Delta t$.

Our numerical equation of motion will tell us $\theta(t_{n+1})$ given $\theta(t_n)$ and $\theta(t_{n-1})$.

Future $\leftarrow$ (Present & past)

We need a way to calculate $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ approximately, given $\theta(t_n)$.
**FIRST DERIVATIVE**

By definition \( \frac{d\theta}{dt} = \lim_{\epsilon \to 0} \frac{\theta(t+\epsilon) - \theta(t)}{\epsilon} = \frac{\text{Rise}}{\text{Run}} \)

Approximately \( \frac{d\theta}{dt} \bigg|_{t_n} \approx \frac{\theta(t_n + \delta t) - \theta(t_n)}{\delta t} = \frac{\theta_{n+1} - \theta_n}{\delta t} \)

Q: At what time is this \( \frac{d\theta}{dt} \)? \( t_n \) ? \( t_{n+1} \) ?

A: At the midpoint, \( t_n + \frac{\delta t}{2} \).

Similarly, \( \frac{d\theta}{dt} \bigg|_{t_{n-1}} \approx \frac{\theta_n - \theta_{n-1}}{\delta t} = \frac{\text{Rise}}{\text{Run}} \)
SECOND DERIVATIVE

By definition \( \frac{d^2 \theta}{dt^2} = \frac{d}{dt} \left( \frac{d \theta}{dt} \right) \)

Approximately \( \frac{d^2 \theta}{dt^2} \approx \frac{d \theta}{dt} \frac{(t + \frac{\Delta t}{2}) - d \theta}{dt} \frac{(t - \frac{\Delta t}{2})}{\Delta t} \) Rise

\( \frac{d^2 \theta}{dt^2} \approx \frac{(\theta_{n+1} - \theta_n)}{\Delta t} - \frac{(\theta_n - \theta_{n-1})}{\Delta t} \) Run

\( \frac{d^2 \theta}{dt^2} \approx \frac{(\theta_{n+1} - \theta_n) - (\theta_n - \theta_{n-1})}{\Delta t^2} \)

\( \frac{d^2 \theta}{dt^2} = \frac{\theta_{n+1} + 2\theta_n + \theta_{n-1}}{\Delta t^2} \)

We'll use this again

Our numerical equation of motion is

\( \frac{\theta_{n+1} - 2\theta_n + \theta_{n-1}}{\Delta t^2} \approx \frac{d^2 \theta}{dt^2} = -\frac{g}{L} \sin \theta \)

\( \theta_{n+1} = (\Delta t)^2 \left( \frac{g}{L} \sin \theta \right) + 2\theta_n - \theta_{n-1} \)

Future \( \leftrightarrow \) Present and Past
\[
\frac{d\theta}{dt} (t_n + \frac{st}{2}) \approx \frac{\theta_{n+1} - \theta_n}{st} \\
\frac{d\theta}{dt} (t_n - \frac{st}{2}) \approx \frac{\theta_n - \theta_{n-1}}{st}
\]

\[
\frac{d^2\theta}{dt^2} (t_n) = \frac{d}{dt} \left( \frac{d\theta}{dt} \right)
\]

\[
\approx \frac{\theta_{n+1} - \theta_n}{st} - \frac{\theta_n - \theta_{n-1}}{st}
\]

\[
= \frac{\theta_{n+1} + \theta_{n-1}}{st}
\]

\[
\frac{d^2\theta}{dt^2} \approx \frac{\theta(t+st) - 2\theta(t) + \theta(t-st)}{st^2}
\]

**Formula for Second Derivative**

\[
\frac{d^2\theta}{dt^2} \approx \frac{\theta(t+st) - 2\theta(t) + \theta(t-st)}{st^2}
\]

**Definitions**

- \( \theta_n \): Temperature at time \( t_n \)
- \( \theta_{n+1} \): Temperature at time \( t_{n+1} \)
- \( \theta_{n-1} \): Temperature at time \( t_{n-1} \)
Discrete Approx.

Equation of Motion for Pendulum

\[ \frac{d^2 \Theta}{dt^2} = \frac{\Theta(t+\Delta t) - 2\Theta(t) + \Theta(t-\Delta t)}{\Delta t^2} = -\frac{g}{L} \sin \Theta \]

We already know \( \Theta(t) \) and \( \Theta(t-\Delta t) \):

Solve for \( \Theta(t+\Delta t) \):

\[ \Theta(t+\Delta t) = -\Delta t^2 \frac{g}{L} \sin \Theta + 2\Theta(t) - \Theta(t-\Delta t) \]

• Excellent algorithm: accuracy, Stability, Fidelity

• At \( t=0 \), need \( \Theta(t-\Delta t) = \Theta(-\Delta t) \)?

Second-order equation: have \( \Theta(0) \) and \( \frac{d\Theta}{dt}(0) \)

Use these to estimate \( \Theta(t-\Delta t) \).

Problem Set 2
**The Wave Equation for a Stretched String**

**DEMO:** Spring attached to Wall
- Slowly: Rigid Body
- Quickly: Different Parts Move at Different Speeds

We describe the string as lots of small **CHUNKS**, each moving in time.

**Locality:** Each chunk feels forces only from neighboring chunks.

*(We ignore gravity)*

**DEMO:** Ribbon on spring moves up and down

**Transverse Wave**

Chunk velocity is perpendicular to axis of string

$y(x,t)$

Height of chunk at $x$ at time $t$
CHUNK VELOCITY, ACCELERATION

PARTIAL DERIVATIVES

Velocity of chunk at $x_0$:
- Along y direction transverse
- Given by $\lim_{\delta t \to 0} \frac{y(x_0, t_0 + \delta t) - y(x_0, t_0)}{\delta t}$
- This is the definition of $\frac{dy}{dt}$ = "partially by partially" $t$

$\frac{dy}{dt}$: Take derivative with respect to $t$.

Acceleration of chunk at $x_0$:
- $\lim_{\delta t \to 0} \frac{\frac{dy}{dt}(x_0, t_0 + \delta t) - \frac{dy}{dt}(x_0, t_0)}{\delta t} = \frac{d^2y}{dt^2}$

So Far: $F = ma$ for chunk, $a = \frac{d^2y}{dt^2}$
CHUNK MASS

\[ \mu = \frac{\text{mass}}{\text{length}} \text{ for string} \]

\[ \delta x = \text{length of string chunk} \]

\[ m = \mu \delta x \]

DIRECTION OF FORCE ON CHUNK

Exercises: Is net force on chunk upward (positive) or downward (negative)? vote, talk, vote

F happens when string is bent \[ \Rightarrow \text{probably second derivative} \]

Second derivative of what?
Second derivative of \( y(x, t) \) with respect to \( x \) at fixed time \( \frac{\partial^2 y}{\partial x^2} \)

More on Thursday
CHUNK FREE BODY DIAGRAM

- Derivation differs from text (same small $\theta$)
- Perfectly transverse motion
  
  No velocity along $x$ $\Rightarrow$ no acceleration along $x$ $\Rightarrow$ no force along $x$
  
  $\Rightarrow x$ component of $\tau_1 = x$ component of $\tau_2 = \tau$

Exercise: What is $y$ component of $\tau_1$, in terms of $\tau$ and $\theta$?

Net Force $= \tau \left[ \tan \theta_2 - \tan \theta_1 \right]$

What is $\tan \theta$ in terms of the height $y(x)$?

$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\Delta y}{\Delta x} = \frac{\partial y}{\partial x}$

$\theta_1$ at $x_0 - \frac{\Delta x}{2}$  $\theta_2$ at $x_0 + \frac{\Delta x}{2}$

Force $= \tau \left[ \frac{\partial y}{\partial x} (x_0 + \Delta x) - \frac{\partial y}{\partial x} (x_0) \right]$
EQUATION OF MOTION
FOR STRETCHED STRING

\[ ma = F \]

\[ a = \frac{\partial^2 y}{\partial t^2}, \quad m = \mu \delta x, \quad F = \tau \left[ \frac{\partial y}{\partial x} (x_0 + \delta x) - \frac{\partial y}{\partial x} (x_0) \right] \]

\[ \int_0^\infty \frac{m}{\mu} \frac{\partial^2 y}{\partial t^2} = \tau \left[ \frac{\partial y}{\partial x} (x_0 + \delta x) - \frac{\partial y}{\partial x} (x_0) \right] \]

\[ \frac{\partial^2 y}{\partial t^2} = \frac{\tau}{\mu} \left[ \frac{\partial y}{\partial x} (x_0 + \delta x) - \frac{\partial y}{\partial x} (x_0) \right] \frac{1}{\delta x} \]

Exercise! What is this as \( \delta x \to 0 ? \)

A: \( \frac{\partial^2 y}{\partial x^2} (x_0) \)

Wave Equation for String:

\[ \frac{\partial^2 y}{\partial t^2} = \frac{\tau}{\mu} \frac{\partial^2 y}{\partial x^2} \]

General wave equation \( \frac{\partial^2 y}{\partial t^2} = \nu^2 \frac{\partial^2 y}{\partial x^2} \)

\( \nu = \sqrt{\frac{\tau}{\mu}} \quad \text{Discuss Tuesday} \)
RESONANCE

DEMO: Hold spring attached to wall
Oscillate small distance vertically, starting
at low frequency; slowly increase
See fundamental, first harmonic, second harmonic
Graph on board

BIG WAVES: STANDING WAVES

![Node and Antinode Diagram]

Q: How many antinodes? \( n = 3 \)

- Come from small forcing ("resonant" = echoing)
- Come at special forcing frequencies
- [Frequencies where string would wiggle by itself]
- Zero at both ends
  (wall end fixed, hand end tiny motions)

Fixed boundary conditions

SERWAY: Free boundary conditions [aluminum rod]

We'll use the wave equation to understand
what these big waves are.

These resonant waves are also called

STANDING WAVES

because they wiggle in place (stand still)
PHYSICS 214 METHOD FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS.

GUESS

Big waves look like sine waves.

\[ y(x,t) = A \sin \left( \frac{2\pi}{\lambda} x \right) \]

Exercise 1: If we want wavelength \( \lambda_1 \), what do we fill into the blank? (A: \( \sin \left( \frac{2\pi x}{\lambda} \right) \) has wavelength \( \lambda \).

\[ \text{At } x = \lambda, \sin \left( \frac{2\pi \lambda}{\lambda} \right) = \text{one wavelength} \]

Exercise 2: Fill in the table

<table>
<thead>
<tr>
<th>Antinodes</th>
<th>( \lambda )</th>
<th>( A ):</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \lambda_1 = 2L )</td>
<td>( \lambda_2 = \frac{2L}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \lambda_2 = \frac{2L}{2} )</td>
<td>( \lambda_3 = \frac{2L}{3} )</td>
</tr>
<tr>
<td>3</td>
<td>( \lambda_3 = \frac{2L}{3} )</td>
<td>( \frac{L}{n} = \text{one another} )</td>
</tr>
<tr>
<td>n</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Amplitude of standing waves oscillates positive and negative in time; frequency $f$

Guess $A(t) = A \sin (2\pi ft)$

Standing wave periodic in time: force at same frequency $\Rightarrow$ resonance

Try solution $y(x,t) = A \sin (2\pi ft) \sin \left( \frac{2\pi x}{l} \right)$

In wave equation $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2}{\partial t^2} \left( A \sin \left( 2\pi ft \right) \sin \left( \frac{2\pi x}{l} \right) \right)$$

Constant treat as constant for $\frac{\partial}{\partial t}$ partial derivative

$$= A \sin \left( \frac{2\pi x}{l} \right) \frac{\partial^2}{\partial t^2} \left( \sin (2\pi ft) \right)$$

$$= A \sin \left( \frac{2\pi x}{l} \right) \frac{d}{dt} \left( 2\pi f \cos (2\pi ft) \right)$$

$$= A \sin \left( \frac{2\pi x}{l} \right) \left( -(2\pi f)^2 \sin (2\pi ft) \right)$$

$$= -(2\pi f)^2 y(x,t) \text{ Simple Harmonic Motion in Time}$$
\[
\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2}{\partial x^2} \left( A \sin \left( 2\pi f t \right) \sin \left( \frac{2\pi x}{\lambda} \right) \right)
\]

\[
\uparrow \quad \text{constant for } \frac{\partial}{\partial x}
\]

\[
= A \sin \left( 2\pi f t \right) \frac{\partial^2}{\partial x^2} \left( \sin \left( \frac{2\pi x}{\lambda} \right) \right)
\]

As before Two derivatives of \( \sin(kx) = -k^2 \sin(kx) \)

\[
= A \sin(2\pi f t) \left( -\left( \frac{2\pi}{\lambda} \right)^2 \sin \left( \frac{2\pi x}{\lambda} \right) \right)
\]

\[
= -\left( \frac{2\pi}{\lambda} \right)^2 \gamma(x, t)
\]

Plug In to Wave Equation

\[
\frac{\partial^2 y}{\partial x^2} = v^2 \frac{\partial^2 y}{\partial x^2}
\]

If standing wave \( y(x, t) = A \sin(2\pi f t) \sin \left( \frac{2\pi x}{\lambda} \right) \)

\[
-\left( 2\pi f \right)^2 y(x, t) = v^2
\]

\[
+ (2\pi f)^2 = +v^2 \left( \frac{2\pi}{\lambda} \right)^2
\]

\[
v^2 = \frac{v^2}{\lambda^2}
\]

\[
\frac{v}{\lambda} = \sqrt{v^2}
\]

\[
\lambda_n = \frac{2\lambda}{n} \quad n = \# \text{ of antinodes}
\]

Resonance \( f_n \lambda_n = v \)

\[
f_n = \frac{v}{\lambda_n} = \frac{n v}{2L}
\]

at special frequencies
Standing Waves Solve the Wave Equation

For n antinodes
\[ \lambda_n = \frac{2L}{n} \]

Trial Solution
\[ y(x,t) = A \sin \left( \frac{2\pi ft}{L} \right) \sin \left( \frac{2\pi x}{\lambda} \right) \]

Sine wave shape
Oscillates in time

Wave equation \( \frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \); Plug in

\[ \frac{\partial y}{\partial t} = \frac{\partial}{\partial t} \left( A \sin \left( \frac{2\pi ft}{L} \right) \sin \left( \frac{2\pi x}{\lambda} \right) \right) \]

Constant

\[ = A \sin \left( \frac{2\pi x}{\lambda} \right) \frac{2}{2f} \sin \left( \frac{2\pi ft}{L} \right) \cos \left( \frac{2\pi ft}{L} \right) \]

\[ = 2\pi f A \sin \left( \frac{2\pi x}{\lambda} \right) \cos \left( \frac{2\pi ft}{L} \right) \]

\[ \frac{\partial^2 y}{\partial t^2} = 2\pi f A \sin \left( \frac{2\pi x}{\lambda} \right) \frac{2}{2f} \cos \left( \frac{2\pi ft}{L} \right) \]

\[ = -(2\pi f)^2 A \sin \left( \frac{2\pi x}{\lambda} \right) \sin \left( \frac{2\pi ft}{L} \right) \]

\[ = -(2\pi f)^2 y(x,t) \]
\[
\frac{\partial^2 X}{\partial x^2} \frac{\partial^2}{\partial x^2} \left( A \sin \left( 2\pi f t \right) \sin \left( \frac{2\pi x}{\lambda} \right) \right) \\
= A \sin \left( 2\pi f t \right) \frac{\partial^2}{\partial x^2} \left( \sin \left( \frac{2\pi x}{\lambda} \right) \right) \\
= \frac{\partial}{\partial x} \left( \frac{2\pi}{\lambda} \cos \left( \frac{2\pi x}{\lambda} \right) \right) = \left( \frac{2\pi}{\lambda} \right)^2 \sin \left( \frac{2\pi x}{\lambda} \right) \\
= -\left( \frac{2\pi}{\lambda} \right)^2 A \sin \left( 2\pi f t \right) \sin \left( \frac{2\pi x}{\lambda} \right) \\
= -\left( \frac{2\pi}{\lambda} \right)^2 \chi(x,t) \\
\]

Plug into wave equation
\[
\frac{\partial^2 X}{\partial t^2} = v^2 \frac{\partial^2 X}{\partial x^2} \\
\]

where \( \lambda \) is wavelength, \( v \) is velocity,

Plug \( f = \frac{v}{\lambda} \)

Special wavelengths satisfy boundary conditions,

Standing Wave Solutions to Wave Equation
\[
\chi_n(x,t) = A \sin \left( 2\pi f_n t \right) \sin \left( \frac{2\pi x}{\lambda_n} \right) \\
\]

Sonometer Demo
TRAVELING WAVES

Another Family of Solutions to the Wave Equation

Clues: Voice
- Travels undistorted from Mouth to Ear
  [Not quite: nearby bolt crack! Far-away lightning rumble...]
- Pulse on Spring
  - Too Fast
  - Longitudinal Better
  - A bit crummy
  - Any shape
- Pulse on Torsional Wave
  - A bit crummy
  - Backward!
  - Any shape
  - Same speed, any shape Call speed $c$.
  - Why?

How to write a solution $y(x, t)$ which can have any shape at all?

$y(x) = y(x, 0) = f(x)$

$y(x, t) = f(x - ct)$

If Pulse moves right, have to move $-ct$ to find where it was $x - ct$ to $x + ct$.
Travelling Waves: Solve the Wave Equation

\( y(x, t) = \) 

Does \( f(x - vt) \) solve the wave equation

\[ \frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \]

We need a notation for derivatives of \( f \) with respect to its argument. Let the slope of \( f \) be \( f' \), and the second derivative be \( f'' \) [Newton's notation]. (So, if \( z = x - ct \), then \( f' = \frac{df}{dz} \) and \( f'' = \frac{d^2 f}{dz^2} \)).

\[ \frac{\partial y}{\partial t} = \frac{d}{dt} \left[f(x - vt)\right] \]

\[ = f'(x - vt) \frac{d}{dt}(x - vt) \]

\[ = f'(x - ct) \left[-v\right] \]

\[ \frac{\partial^2 y}{\partial t^2} = \frac{d}{dt} \left[-vf'(x - vt)\right] = -v \frac{d}{dt} f'(x - vt) \]

\[ = -v f''(x - vt) \frac{d}{dt}(x - vt) \]

\[ = v^2 f''(x - vt) \]

\[ \frac{\partial y}{\partial x} = \frac{d}{dx} \left[f(x - vt)\right] = f'(x - vt) \frac{d}{dx}(x - vt) = f'(x - vt) \]

\[ \frac{\partial^2 y}{\partial x^2} = \frac{d}{dx} f'(x - vt) = f''(x - vt) \]
Does \( y(x, t) = f(x-ct) \) satisfy the wave equation \( \frac{\partial^2 y}{\partial t^2} = \frac{1}{\mu} \frac{\partial^2 y}{\partial x^2} = \nu^2 \frac{\partial^2 y}{\partial x^2} \)?

\[
\frac{\partial y}{\partial t} = -c f'(x-ct) \quad \frac{\partial y}{\partial x} = f'(x-ct)
\]

\[
\frac{\partial^2 y}{\partial t^2} = c^2 f''(x-ct) \quad \frac{\partial^2 y}{\partial x^2} = f''(x-ct)
\]

**Traveling Wave** \( y(x, t) = f(x-ut) \) satisfies wave equation for any shape \( f \).

\( V = \) pulse velocity = velocity in \( x \) direction of shape.

**Questions #1:** Are these traveling waves?

If so, what is \( V \)?

\[
y(x, t) = \frac{2}{(x-4t)^2+1}
\]

\[
y(x, t) = (x^2+t^2)
\]

\[
y(x, t) = \cos(6x + 2t)
\]

\[
y(x, t) = e^{-(x+8t)^6+37}
\]
Question #2: Pulse with funny shape at \( t=0 \) travels to right velocity \( v = \sqrt{\frac{F}{m}} \).

Draw \( y(0,t) \), the height of the chunk at \( x=0 \).

Question #3: Is this a traveling wave?

"3D" plots of \( y(x,t) \) A standing wave.

Notice shape flips over!
SUPERPOSITION

Suppose \( y_1(x,t) \) and \( y_2(x,t) \) solve the wave equation.

So will \( y(x,t) = A y_1(x,t) + B y_2(x,t) \).

**Proof:**

\[
\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}
\]

\[
\frac{\partial y}{\partial t} = \frac{\partial}{\partial t} \left[ A y_1 + B y_2 \right] = A \frac{\partial}{\partial t} (y_1(x,t)) + B \frac{\partial}{\partial t} (y_2(x,t))
\]

\[
\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t} \left[ A \frac{\partial y_1}{\partial t} + B \frac{\partial y_2}{\partial t} \right] = A \frac{\partial}{\partial t} \left( \frac{\partial y_1}{\partial t} \right) + B \frac{\partial}{\partial t} \left( \frac{\partial y_2}{\partial t} \right)
\]

\[
= A \frac{\partial^2 y_1}{\partial t^2} + B \frac{\partial^2 y_2}{\partial t^2} = A v^2 \frac{\partial^2 y_1}{\partial x^2} + B v^2 \frac{\partial^2 y_2}{\partial x^2}
\]

use wave equation

\[
= \left[ \text{same things backwards} \right] v^2 \frac{\partial^2 (A y_1)}{\partial x^2} + v^2 \frac{\partial^2 (B y_2)}{\partial x^2}
\]

\[
= v^2 \frac{\partial^2}{\partial x^2} (A y_1 + B y_2(x,t)) = v^2 \frac{\partial^2 y}{\partial x^2}
\]

\[
\frac{\partial y}{\partial x}, \ \frac{\partial^2 y}{\partial x^2}, \ y \ \text{are linear} \Rightarrow \text{solutions add.}
\]
Examples of Superposition:

1. \[ A \sin \left( \frac{2\pi x}{\lambda} \right) \sin \left( 2\pi ft \right) \]
   - Standing wave amplitude
   - Big waves \( \propto \) small waves

2. \[ \sin (x - vt) = \sin(x) \cos(vt) - \cos(x) \sin(vt) \]
   - Angle addition formula
   - \( \sin(A + B) = \sin A \cos B + \cos A \sin B \)
   - \( \sin(A - B) = \sin A \cos B - \cos A \sin B \)
   - Traveling sine wave is superposition of two standing waves.

3. \[ \sin \left( \frac{2\pi x}{\lambda} \right) \cos \left( 2\pi ft \right) \]
   - \( = \sin \left( \frac{2\pi x + 2\pi ft}{\lambda} \right) + \sin \left( \frac{2\pi x - 2\pi ft}{\lambda} \right) \)
   - Velocity \( v = \pm \frac{1}{L} \)
   - Standing wave is superposition of two traveling sine waves.

4. Draw \( y(x, \pm t) \)

5. Harmonics & Overtones
Is this a traveling wave?  
A standing wave?  
Is it moving right? Left?  
Notice that the shape $y(x)$ flips over as $y(t)$!
Is this a traveling wave?
A standing wave?
What kind of boundary condition does it have?
What kind of wave is this? We call this a "plane wave"
**ENERGY AND POWER**

for Waves on Strings

Why study energy & power in waves?

- Direct applications
  - Microwaves  Sound (db)
  - Sunlight  Transmission Lines
- Conservation Laws
  - Reflection & Transmission
  - Boundary Conditions
- Scaling: Fix shape, change height Δy & width Δx
- Cool equations

Quick Review: Energy & Power in Elevators

\[ W = \text{Energy of elevator} = mg \text{h} \]

\[ \text{Work done by cable on elevator} = mg \Delta h = F \text{d}h \]

\[ F = -T = -mg \]

Power upward = Positive = \( mg \frac{\Delta h}{\Delta t} = F \cdot W \)

Power downward = Negative = \( mg \)
Power for Waves on Strings

\[ F = -P \tan \theta \]
\[ = -P \frac{dy}{dx} \]

Q: Is the power positive (flowing right) or negative (flowing left)?

Is the work done by the left-hand segment ("elevator") positive or negative?

Energy is flowing left, into the elevator.

\[ \text{Power} = F \cdot \Delta x = -P \frac{dy}{dx} \frac{dy}{dt} \]

\[ \frac{dy}{dx} > 0, \frac{dy}{dt} > 0 \text{ [picture shown]} \]

\[ \rightarrow \text{Power} < 0 \text{ [energy flowing left, into "elevator"]} \]

\[ \text{Power} = -P \frac{dy}{dx} \frac{dy}{dt} \]

Demo: Wave on Spring

\[ \cdot \text{Estimate } \frac{F}{x} \approx 1-10 \text{ N} \]
\[ \cdot \text{Estimate } \frac{dy}{dx} \approx 0.1-0.3 \]
\[ \cdot \text{Estimate } \frac{dy}{dt} \approx 0.1 \text{ m/s} \]

\[ \rightarrow \text{Prefer electrical power [bigger pipe]} \]
Energy for Waves on Strings

\[ \text{KE} = \text{Kinetic Energy} = \frac{1}{2}m \left( \frac{\partial y}{\partial t} \right)^2 \]

\[ = \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 \delta x \]

Total KE = \sum_{\text{segments}} \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 \delta x = \int \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 \, dx \]

KE Density/Length = \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2
Potential energy for horizontal $(\Delta = 0)$ is zero (defined).

Integrate force $\cdot d\Delta$ from $\Delta = 0$ to $\Delta = \frac{\partial y}{\partial x} \delta x$ to get work done.

$\Rightarrow$ Potential energy

$\text{Force} = \tau \frac{\partial}{\partial x} \frac{\Delta}{\delta x}$

$\text{Force to pull}$

$\text{chunk upward} = \tau \frac{\Delta}{\delta x}$

$\sum (\text{Force}) \, d\Delta = \sum \left( \frac{\partial y}{\partial x} \delta x \right) \tau \frac{\Delta}{\delta x} \, dx$

$= \frac{\tau}{\delta x} \left( \frac{\Delta^2}{2} \right) \frac{\partial y}{\partial x} \delta x = \frac{\tau}{\delta x} \left( \frac{\partial y}{\partial x} \delta x \right)^2$

$= \frac{1}{2} \tau\left( \frac{\partial y}{\partial x} \right)^2 \delta x$
Total Energy Density

\[ u(x) = \frac{1}{2} \mu \left( \frac{\partial^2 x}{\partial t^2} \right)^2 + \frac{1}{2} \rho \left( \frac{\partial x}{\partial t} \right)^2 \]

Total energy in string

\[ U = \int_0^L u(x) \, dx = \int_0^L \frac{1}{2} \mu \left( \frac{\partial^2 x}{\partial t^2} \right)^2 + \frac{1}{2} \rho \left( \frac{\partial x}{\partial t} \right)^2 \, dx \]

Exercise: Total energy of string w/sine wave.
BOUNDARY CONDITIONS AND ENERGY CONSERVATION

What is the equation of motion for the end of the string? Depends on Boundary Condition.

Fixed Boundary Condition
\( y(0,t) = 0 \)

\( x = 0 \)

(Guitars, Springs at Wall)

Q: Does energy flow into or out of a fixed boundary?

A: Energy flow = Power = \(-\tau \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}\)

But \(\frac{\partial y}{\partial t} = 0\) at a fixed boundary

\( \Rightarrow \) Energy is conserved.

How to Solve Wave Equation For Fixed Boundary Conditions

DEMO: Pulse on spring inverts at wall.

Use SUPERPOSITION.
**Superposition**

Suppose \( y_1(t) \) and \( y_2(t) \) satisfy the wave equation. Then \( y_1(t) + y_2(t) \) also solves the equation.

Proof:

\[
\frac{\partial^2 y_1}{\partial t^2} = v^2 \frac{\partial^2 y_1}{\partial x^2} \quad \frac{\partial^2 y_2}{\partial t^2} = v^2 \frac{\partial^2 y_2}{\partial x^2}
\]

Use wave eqn

\[
\frac{\partial^2}{\partial t^2} (y_1 + y_2) = \frac{\partial^2 y_1}{\partial t^2} + \frac{\partial^2 y_2}{\partial t^2} = v^2 \frac{\partial^2 y_1}{\partial x^2} + v^2 \frac{\partial^2 y_2}{\partial x^2}
\]

\[
= v^2 \left( \frac{\partial^2 y_1}{\partial x^2} + \frac{\partial^2 y_2}{\partial x^2} \right) = v^2 \frac{\partial^2 (y_1 + y_2)}{\partial x^2}
\]

\( y_1 + y_2 \) is solution

---

\( t=0 \)

Fictitious Pulse

\( \text{Pulse starts moving left} \)

\( f(x + vt) \)

\( \text{Pulse leaves moving right} \)

\( \text{flipped, inverted} \)

\( f(-(-x - vt)) \)
GUESS Solution \( f(x+vt) - f(-(x-vt)) \)

1. Solves Wave Equation?
   - \( f(x+vt) \) solves wave equation \( \frac{\text{shape}}{z} = f(-z) \)
   - \( -f(-(x-vt)) \) solves wave equation \( \frac{\text{shape}}{z} = x-vt \)

   \( \Rightarrow \) [Superposition] \( f(x+vt) - f(-(x-vt)) \) solves wave equation

2. Satisfies initial condition?

   \[ f(x+vt) = 0 \text{ for } x<0, \ t=0 \]
   \( \Rightarrow \) \( f(-(x-vt)) = 0 \) for \( x>0 \) at \( t=0 \)

   \( \Rightarrow \) at \( t=0 \), "fictional pulse" not on string yet.

3. Satisfies boundary condition?

   \[ \gamma(0,t) = f(x+vt) - f(-(x-vt)) \text{ at } x=0 \]
   \[ = f(vt) - f(-(-vt)) \]
   \[ = f(vt) - f(vt) = 0 \]
What other boundary condition conserves energy?

Power = -ρ \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}

\frac{\partial y}{\partial t} = 0 \Rightarrow \text{Power} = 0 \quad \text{Fixed Boundary Condition}

\frac{\partial y}{\partial x} = 0 \Rightarrow \text{Power} = 0 \quad \text{?}

γ \frac{\partial y}{\partial x} = \text{Vertical component of force on string}

\frac{\partial y}{\partial x} = F_y \quad \text{ZERO FORCE} \Leftrightarrow \frac{\partial y}{\partial x} = 0 \Rightarrow \text{FREE BOUNDARY}

\text{Examples of sound, transmission lines, ...}

\text{Superposition}

f(x+vt) + f(-(x-vt)) \Rightarrow \text{not inverted, flipped right}

\text{DEMO: TORSIONAL WAVE MACHINE}
SCALING

What happens to the power, energy density, and total energy of a pulse when its size changes?

TRAVELING

Same Shape

Reference Pulse

Pulse Amplitude Changed

\[ \gamma_2 = A \gamma_1 \]

\[ P_2 = -A^2 \frac{\partial \gamma_2}{\partial x} \frac{\partial \gamma_2}{\partial t} \]

\[ = -A^2 \frac{\partial (A \gamma_1)}{\partial x} \frac{\partial (A \gamma_1)}{\partial t} = A^2 \frac{\partial \gamma_1}{\partial x} \frac{\partial \gamma_1}{\partial t} \]

\[ = A^2 P_1 \]

\[ u_1(x) = \frac{1}{2} \mu \left( \frac{\partial \gamma_1}{\partial t} \right)^2 + \frac{1}{2} \gamma_1 \left( \frac{\partial \gamma_1}{\partial x} \right)^2 \]

\[ u_2(x) = \frac{1}{2} \mu \left[ \frac{\partial (A \gamma_1)}{\partial t} \right]^2 + \frac{1}{2} A^2 \left( \frac{\partial (A \gamma_1)}{\partial x} \right)^2 \]

\[ = A^2 u_1(x) \]

\[ (\text{Total Energy})_1 = \int u_1(x) \, dx \quad (\text{Total Energy})_2 = \int A^2 u_1(x) \, dx \]

\[ = A^2 (\text{Total Energy})_1 \]

Power, Energy Density, and Total Energy

All Scale with the Square of the Amplitude (Fixed Shape)
Reference Pulse

Shape \( y_1(x,t) = f(x-ut) \)

Slope \( \frac{\partial y_1}{\partial x} (x,t) = f'(x-ut) \)

\( f'(z) = \frac{df(z)}{dz} \)

Chunk Velocity \( \frac{\partial y_1}{\partial t} = -v f'(x-ut) \)

Pulse Width Changed

\( y_2(x,t) = \frac{1}{\Delta x} \int f'(x-ut) \, dx \)

Slope \( \frac{\partial y_2}{\partial x} (x,t) = \frac{1}{\Delta x} f'(x-ut) \)

\( \frac{\partial y_2}{\partial t} (x,t) = -v \frac{1}{\Delta x} f'(x-ut) \)

\[ P_1 = \text{Power} = -r \frac{\partial y_1}{\partial x} \frac{\partial y_1}{\partial t} \]

\[ P_2 = \text{Power}_2 = -r \frac{\partial y_2}{\partial x} \frac{\partial y_2}{\partial t} \]

\[ = -\frac{1}{(\Delta x)^2} P_1 \]

Energy Density \( u_1(x,t) = \frac{1}{2} \rho \left( \frac{\partial y_1}{\partial t} \right)^2 + \frac{1}{2} \rho u \left( \frac{\partial y_1}{\partial x} \right)^2 \)

\[ = \frac{1}{2} \rho \left( f'(z) \right)^2 + \frac{1}{2} \rho \left( \frac{df(z)}{dz} \right)^2 \]

\[ = \frac{1}{2} \rho \left( f'(z) \right)^2 + \frac{1}{2} \rho \left( \frac{df(z)}{dz} \right)^2 \]

\[ = \frac{1}{(\Delta x)^2} u_1 \left( \frac{x}{\Delta x}, t \right) \]

\[ \Rightarrow u_2 \Delta x + t \]

\[ \Rightarrow u_1 \Delta x + \frac{1}{\Delta x} \]
\[(\text{Total Energy})_2 = \int u_1(x,t) \, dx\]

\[u_1(x)\]

\[\downarrow \text{Down by } \frac{\Delta x}{2}\]

\[\leftarrow \text{Stretched by } \Delta x \rightarrow\]

\[u_2(x)\]

\[\text{Integral} = \text{Area underneath}\]
\[
\frac{1}{\Delta x} = \frac{\Delta x}{(\Delta x)^2} = \text{Height}
\]

\[\text{(Power)}_2 = \frac{A^2}{\Delta x^2} \left(\text{Power}\right)_1\]

\[U_2 = \frac{A^2}{\Delta x^2} \, U_1\]

\[\text{(Total Energy)}_2 = \frac{A^2}{\Delta x} \left(\text{Total Energy}\right)_1\]

\[\text{Exact if shape of pulse kept fixed}\]

Corresponds to Dimensional Analysis

\[P = -\tau \frac{\partial u}{\partial x} \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \quad \partial y \sim A \quad \partial t \sim \Delta t = \Delta x / v \quad \frac{\partial A^2}{\partial x} (\Delta x / v)\]

\[u = \left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2} \tau \left(\frac{\partial x}{\partial x}\right)^2 \sim \frac{1}{2} \mu \left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2} \tau \left(\frac{A^2}{\Delta x^2}\right)^2 \sim \frac{1}{2} \left(\frac{A^2}{\Delta x^2}\right)\]

\[\text{Total Energy} = \int u \, dx \sim u \, \Delta x \sim \tau \left(\frac{A^2}{\Delta x}\right)\]
Reflection and Transmission at Discontinuities in Strings

Two different strings $\mu_1$ & $\mu_2$ tied together:

$\mu_1$ $\mu_2$

$\Delta x_I$ Initial Pulse

**Demo: Computer**

Step Up: Time to Run = 0.1025

- $\mu_2 > \mu_1$
- String continuous at peak
- Reflected pulse inverted
  - $A_R < 0$ Heavy $\mu_2$ like fixed boundary
- Transmitted pulse smaller, narrower $A_T < A_I$, $\Delta x_T < \Delta x_I$
- Pulses all same shape

Reflection here similar to reflection of light off of water - water "thick string" air "thin string".

Some light transmitted into water
Some reflected $\Rightarrow$ see face in paddle
Step Down: Time to Run = 0.04125

- $\mu_2 < \mu_1$
- String continuous
- Reflected pulse not inverted
- Transmitted pulse wider, higher: $\Delta x_T > \Delta x_I$, $A_T > A_I$ not always
- Pulses all same shape

You will verify two laws in Pythag:

\[ A_I + A_R = A_T \]
Continuity of String

\[ P_I + P_R = P_T \]
Power into = Power away
knot = Conservation of Energy

In the homework, you'll use SCALING and these two laws to solve for $A_R$ and $A_T$. 
Discretizing the Wave Equation

How can we solve the wave equation numerically? Can’t store the curve \( y(x) \) [\( \infty \) # points].

**Discretize Space**

Need to sample \( y(x, t) \) at discrete set of positions \( x_n = n \Delta x \).

Approximate First Derivative \( \frac{\partial^2 y}{\partial x^2} (x_n) = \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x^2} \)

Approximate Second Derivative \( \frac{\partial^2 y}{\partial x^2} (x_n) = \frac{\frac{\partial y}{\partial x} (x_n + \frac{\Delta x}{2}) - \frac{\partial y}{\partial x} (x_n - \frac{\Delta x}{2})}{\Delta x} = \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x^2} \)

\( \frac{\partial^2 y}{\partial x^2} (x_n) \) can be approximated as \( \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x^2} \).
Discrete Approximation for Wave Equation

$$\frac{d^2 Y_n}{dt^2} = v^2 \frac{Y_{n+1} + Y_{n-1} - 2Y_n}{\delta x^2}$$

Aside: Sound in Crystals

Chain of Balls and Springs

Atoms and Chemical Bonds

One-dimensional Crystal, Longitudinal wave

$$\delta x = \text{Spring length}$$

$$x_n = n \delta x = \text{Undefomed Position}$$

$$x_{n+1} + u_n = \text{Actual Position}$$

Bond $B$ is stretched: length is $\delta x + u_{n+1} - u_n$

Force on atom is $K(u_{n+1} - u_n)$ to right

Bond $A$ is squeezed: length is $\delta x + u_n - u_{n-1}$

Force on atom is $-K(u_n - u_{n-1})$ to right.

$$ma = m \frac{d^2u_n}{dt^2} = K(u_{n+1} - u_n) - K(u_n - u_{n-1})$$

$$\frac{d^2u_n}{dt^2} = \frac{K}{m} (u_{n+1} + u_{n-1} - 2u_n)$$

Exactly the same equation: $\frac{v^2}{\delta x^2} \Leftrightarrow \frac{K}{m}$

Real chain of atoms $\Leftrightarrow$ Approximate wave equation.
Discretize Time

\[ y_n(t) \text{ must be discretized too: sample at } n \Delta t, \Delta x \]

\[ \frac{\partial^2 y_n}{\partial t^2} \approx \frac{y_n(t+\Delta t) + y_n(t-\Delta t) - 2y_n(t)}{(\Delta t)^2} \]

\[ \approx \frac{v^2}{(\Delta x)^2} \left[ y_{n+1} - 2y_n(t) + y_{n-1} \right] \]

Solve for future from past & present

\[ y_n(t+\Delta t) = 2y_n(t) - y_n(t-\Delta t) + \frac{v^2}{(\Delta x)^2} \left[ y_{n+1}(t) - 2y_n(t) + y_{n-1}(t) \right] \]

Up to you to choose \( \Delta t, \Delta x \)

Not a velocity!
Initial Conditions

First Time Step:

\[ y_n(st) = 2y_n(0) + y_n(-st) + \frac{v^2}{(st)^2} (y_n(0) + y_{n-1}(0) - 2y_n(st)) \]

Need

\[ y_0(0), y_1(0), y_2(0), \ldots, y_N(0) \]

and \[ y_0(-st), y_2(-st), \ldots, y_N(-st) \]

to get started. If we begin with a flat string at rest

\[ y_n(-st) = 0 \]

\[ y_n(0) = 0 \]

Boundary Conditions

Equation of Motion for \( y_0, y_N \):

Forced Boundary Condition \( y_0(t) = y(t) \)

\( y(t) = \) height of human hand at time \( t \).

Fixed Boundary Condition: \( y_N(t) = 0 \)

Free Boundary Condition (Best Version) \( t \) at \( N \):

Zero Slope \( \iff \) Symmetric

\[ y_{n-1} = y_{n-1} \]

\[ \frac{\partial^2 y_n}{\partial x^2} = \frac{y_{n+1} - 2y_n + 2y_{n-1}}{(st)^2} = -2y_{n-1} - 2y_n \]
SOUND

DEMO: ORGAN PIPES
OSCILLOSCOPE & MIKE
TUNING FORK
FREQUENCY ANALYZER

Shape of wave
Harmonics

ORGAN PIPE, FLUTE, TUBA, ...

Atom Displacement $s(x)$

Closed END

$\uparrow$ Displacement from Zero

Pressure $P$
(Density $\rho$)

Pressure Matches Outside

Pressure, Density
Fluctuate About
Atmospheric $P_{atm}$

$P_{atm}$

Free Boundary
(Approximately)

Fixed Boundary
Wave Equation for Sound in One Dimension

\[ \lambda > a, \text{ spacing between atoms} \]

Air, water, solids: Pressure depends on Volume

\[ P = P_0 - B \left( \frac{\Delta V}{V} \right) \]

*B = Bulk Modulus

Good for small \( \frac{\Delta V}{V} \)

\[ \frac{V}{V_0} = A \delta x \]

\[ V + \Delta V = A \left\{ \delta x + s(x + \delta x) - s(x) \right\} \]

\[ \Delta V = A \left[ s(x + \delta x) - s(x) \right] \]

\[ P - P_0 = -B \frac{\Delta V}{V} = -B \frac{A \left[ s(x + \delta x) - s(x) \right]}{A \delta x} = -B \frac{\partial s}{\partial x} \]

Pressure is Force per unit Area

Force = \( AP(x) \)

\[ \Delta x \]

\[ \Delta P(x) \]

\[ \Delta P(x + \delta x) \]

\[ \Delta P(x) - \Delta P(x + \delta x) \]

\[ \Delta P(x) \]

\[ \Delta P(x + \delta x) \]

\[ = m a \]

\[ \rho A \delta x \frac{\partial^2 s}{\partial t^2} \]

\[ \rho A \delta x \frac{\partial^2 s}{\partial t^2} = AP \left( P(x) - P(x + \delta x) \right) \]

\[ -B \frac{\partial s}{\partial x} \]

\[ \frac{\partial s}{\partial x} = \frac{1}{\rho} \frac{P(x) - P(x + \delta x)}{\delta x} \]

\[ = -\frac{1}{\rho} \frac{\partial P}{\partial x} = \frac{B}{\rho} \frac{\partial^2 s}{\partial x^2} \]
Wave Equation for Sound

\[ \frac{\partial^2 s}{\partial t^2} = \frac{B}{\rho} \frac{\partial^2 s}{\partial x^2} \]

Velocity of Sound

in Air, 20°C

= 343 m/s \approx \frac{1}{2} \text{ mile/hour} \quad \text{(Sinusoidal)}

What's the Pressure for Traveling Wave?

\[ s(x, t) = s_{\text{max}} \cos \left( \frac{2\pi x}{\lambda} - 2\pi ft \right) \]

\[ P - P_0 = -B \frac{\partial s}{\partial x} = \frac{2\pi B}{\lambda} s_{\text{max}} \sin \left( \frac{2\pi x}{\lambda} - 2\pi ft \right) \]

\[ \frac{P_{\text{max}}}{P_{\text{max}}} \]

What's the Kinetic Energy Density?

\[ \frac{\text{Kinetic Energy}}{\text{Volume}} = \frac{1}{2} \left( \frac{\partial s}{\partial t} \right)^2 = \frac{1}{2} \rho \left( \frac{\partial s}{\partial t} \right)^2 \]

Potential Energy = Kinetic Energy for Travelling Wave

Total Energy Density = \rho \left( \frac{\partial s}{\partial t} \right)^2

= \rho s_{\text{max}} \left( \frac{\partial s}{\partial t} \right)^2 \sin^2 \left( \frac{2\pi x}{\lambda} - 2\pi ft \right)
What's the **Intensity** of a traveling sound wave?

\[
\text{Intensity} = \frac{\text{Power}}{\text{Area}} = \frac{\text{(Energy Density)}}{\text{Velocity}} = \rho \frac{(\frac{\partial s}{\partial t})^2}{2} = \rho \frac{\sqrt{E_p}}{(\frac{\partial s}{\partial t})^2}
\]

\[
I = \sqrt{E_p} \frac{(\frac{\partial s}{\partial t})^2}{2} \Rightarrow \sqrt{E_p} (2\pi f)^{\frac{3}{2}} s_{\text{max}}^2 \sin^2 \left( \frac{2\pi x}{x - 2\pi f} \right)
\]

What's the **Average Intensity**?

**Useful Trick:** Average of \( \sin^2 \) is \( \frac{1}{2} \)

\[
\sin^2 + \cos^2 = 1
\]

average \( \sin^2 = \frac{1}{2} \)

average \( \cos^2 = \frac{1}{2} \)

\[
\text{Average Intensity} = \sqrt{E_p} (2\pi f)^{\frac{3}{2}} s_{\text{max}}^2 \frac{1}{2}
\]

[Express in terms of \( P_{\text{max}} \)]

**Units**

Intensity = Joules/sec per unit area = Watts/m²

At 1000 Hz, you can hear \( I_0 = 10^{-12} \) W/m² = 1 dB

Corresponding to air displacing \( S_{\text{max}} = 10^{-11} \) m \~\ 30 \text{ atoms}.

A power mower \( I = 10^{-2} \) W/m² = ten Giga \( (I_0) \) ?

**Decibels** \( \beta = 10 \log_{10} \left( \frac{I}{I_0} \right) \)

Lawn Mower = \( 10^{10} I_0 = 100 \) dB
The Wave Equation for Light

We use two of Maxwell's equations, covered last semester:

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Leftrightarrow \quad \int \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} \]

Faraday's Law of Induction

Changing magnetic flux leads to voltage around a loop

How electrical generators work

Ampere's Law [Generalized]

\[ \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \Leftrightarrow \quad \int \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\partial \Phi}{dt} \]

Currents [and changing E-fields] produce B fields around them
How electromagnets work

Try to redo this lecture! Use integral form of Maxwell's equations?

Reminder: What's a curl? Field $\vec{F} = (F_x, F_y, F_z)$

$\nabla \times \vec{F} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$

Want a solution for empty space (no current $j$)

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

How to solve these differential equations?

-Guess-
• Want a plane wave: \( \vec{E} \) and \( \vec{B} \) depend only on \( x \).
  
  Note: they'll point along \( y \) & \( z \).

\[
\frac{\partial \vec{E}}{\partial y} = \frac{\partial \vec{E}}{\partial z} = \frac{\partial \vec{B}}{\partial y} = \frac{\partial \vec{B}}{\partial z} = 0 \quad \text{Much Simpler!}
\]

\[
\nabla \times \vec{E} = (0, -\frac{\partial E_y}{\partial x}, \frac{\partial E_y}{\partial t}) = -\left( \frac{\partial B_x}{\partial t}, \frac{\partial B_x}{\partial t}, \frac{\partial B_x}{\partial t} \right)
\]

\[
\nabla \times \vec{B} = (0, -\frac{\partial B_z}{\partial x}, \frac{\partial B_y}{\partial x}) = \mu_0 \vec{E}_0 \left( \frac{\partial E_z}{\partial t}, \frac{\partial E_z}{\partial t}, \frac{\partial E_z}{\partial t} \right)
\]

• First components

  \( 0 = -\frac{\partial B_x}{\partial t} \quad 0 = \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} \)

  \( E_x, B_x \) constant

  Guess = 0 \quad \vec{E}_x = \vec{B}_x = 0 \quad \text{Transverse Wave.}

• Too many pieces: set one term zero, see if can work

Set \( E_z = 0 \)

Second component of \( \nabla \times \vec{E} \Rightarrow -\frac{\partial E_z}{\partial t} - \frac{\partial B_y}{\partial t} = 0 \)

Set \( B_y = 0 \) too

Guess: \( \vec{E} = (0, E_y, 0) \quad \vec{B} = (0, 0, B_z) \)

\[
\nabla \times \vec{E} = (0, 0, \frac{\partial E_y}{\partial x}) = -(0, 0, \frac{\partial B_z}{\partial t}) = -\frac{\partial \vec{B}}{\partial t}
\]

\[
\nabla \times \vec{B} = (0, -\frac{\partial B_z}{\partial x}, 0) = (0, \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t}, 0) = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}
\]
\[
\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}, \quad \frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}
\]

* Try for Wave Equation *

\[
\frac{\partial^2 E_y}{\partial x^2} = \frac{\partial}{\partial x} \left( -\frac{\partial B_z}{\partial t} \right) = \frac{\partial}{\partial t} \left( -\frac{\partial B_z}{\partial x} \right) = \frac{\partial}{\partial t} \left( \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \right)
\]

Exchange Orders of Differentiation

\[
\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}
\]

\[
\frac{\partial^2 E_y}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 E_y}{\partial x^2}
\]

Wave Equation for Electromagnetic Radiation

\[
C = \sqrt{\frac{1}{\mu_0 \varepsilon_0}} = \text{Speed of Light}
\]
Index of Refraction & Snell's Thin Films

Light in glass travels more slowly, say $v = \frac{3}{4} c$.

[Different kinds of glass have different $v$].

600 nm

- Definition: The index of refraction of a material $n$ is $c/v$.

Note: in 214, $n > 1$ [Einstein].

Q1: What is $n$ for our glass? $\frac{v}{c} = 1.33$

Q2: What is $v$ for our glass? $\frac{3}{4} c = 2.25 \times 10^8$ m/s

Q3: Is $\lambda'$ inside the glass larger or smaller than $\lambda$?

Q4: What is $\lambda'$ inside the glass? $\frac{\lambda'}{c} = \frac{3}{4} = \frac{3}{4} \lambda_f$

$\lambda' = \frac{3}{4} \lambda = 450$ nm

Q5: What color is the light inside the glass? The same color as it was outside.
Snell's Law

- How much does light bend [refract] as it goes from water to glass?
  - Measure $\theta_1$, $\theta_2$ from the normal to the surface.

- Wave peaks must match on the surface.

- Hypotenuse = Distance between wave fronts on the surface
  \[ \frac{\lambda_{\text{vac}}}{\sin \theta_1} = \frac{\lambda_{\text{vac}}}{\sin \theta_2} \]

- \[ \lambda_1 = \frac{\lambda_{\text{vac}}}{n_1}, \quad \lambda_2 = \frac{\lambda_{\text{vac}}}{n_2} \]

- \[ \frac{\lambda_{\text{vac}}}{n_1 \sin \theta_1} = \frac{\lambda_{\text{vac}}}{n_2 \sin \theta_2} \]

- \[ n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad \text{Snell's Law} \]
Thin Films:

Vacuum

Glass $n_2 = \frac{4}{3}$

Vacuum

$\rightarrow 600 \leftarrow$ nm

Vacuum

What is the phase difference between the light which passes through the glass and the light which passes around the glass?

1.5 wiggles = $3\pi$ in vacuum

$\lambda = (600) \frac{3}{4} = 450$ nm in side glass

2 wiggles = $4\pi$ in glass

$\phi_{\text{glass}} - \phi_{\text{vacuum}} = \pi \quad \left[ \frac{1}{2} \text{ wiggle} \right]$

$\Rightarrow$ Destructive Interference
Phase Shifts on Reflection

Notice: 214 курс
Picture shows light at small angle only for clarity.

Any time a wave reaches a discontinuity in the velocity of propagation, part of the wave is reflected.

Glass has lower sound velocity → heavier string

Pulse goes from light string to heavy string; reflected pulse inverted.

Incident

Inverted sine wave =

Reflected pulse

Pulse goes from heavy string to light string; no inversion; no phase shift
"Reflectionless" Coatings

Path B \[ \rightarrow \]
Path A

Air \[ n_0 \approx 1 \]
Coating \[ n_1 \]
Glass \[ n_2 \]

Distance \( 2L \) \[ \frac{2\pi}{\lambda} \text{ wiggles} \]
One reflection from \( \frac{2\pi}{\lambda} \)
high to low \( V \)

\( \phi_A = \) Phase for path A & One reflection from \( \frac{2\pi}{\lambda} \)
low \( V \) to lower \( V \)
Distance \( 2L \) \[ \frac{2\pi}{\lambda} \text{ wiggles} \] \[ \frac{2\pi}{\lambda} \]

\( \lambda' = \frac{\lambda}{n_1} \)

Phase difference \( = \phi_A - \phi_B = \frac{4\pi L}{\lambda} + \frac{2\pi}{\lambda} \)
\( - \left( \frac{4\pi L}{\lambda} + \frac{2\pi}{\lambda} + \frac{4\pi n t}{\lambda} \right) \)
\( = - \frac{4\pi n t}{\lambda} \)

Destructive interference reduces reflections

\( \frac{4\pi n t}{\lambda} = \pm \pi, \pm 3\pi, \ldots \); \( \frac{n t}{\lambda} = \frac{n}{\lambda} = \pm \frac{1}{4}, \pm \frac{3}{4}, \ldots \)

"Quarter wave plate" reduces reflections.
Phasors are our geometrical method for adding the amplitude of different waves.

**Phasor with one wave source**

\[ E(t) = E_0 \sin(\omega t) \]

**Q:** What is \( E(t) \)?

**A:** \( E(t) = E_0 \sin(\omega t) \)

- The phasor for the wave \( E_0 \sin(\omega t) \) has length \( E_0 \), angle \( \omega t \).
- The vertical component of the phasor is \( E(t) \).

**Q:** What is the intensity \( I(t) \)?

**A:** \( I(t) = \epsilon_0 c E_0^2 \sin^2 \omega t \) [EM wave reading]

- The intensity of a wave is proportional to the amplitude squared.
- We won't care much about the constant \( E_0 \) in this part of 214; we'll ask questions about intensity ratios.

**Q:** What is the average intensity \( I_{\text{avg}} \) over time?

\[ I_{\text{avg}} = \frac{\epsilon_0 c E_0^2}{2} \cdot \sin^2 \omega t \]

- The average intensity of a wave is proportional to the square of the phasor's length.
Phasor with two wave sources

Half the light comes through glass

\[ E_d(t) = E_0 \sin(wt + \phi) \]

\[ E_p(t) = E_0 \sin(wt) \]

\[ E(t) = E_1(t) + E_2(t) = E_0 \sin wt + E_0 \sin(wt + \phi) \]

- The relative angle between the two phasors remains at \( \phi \).
- The whole triangle rotates at frequency \( wt \).
- \( E_R \), the length of the resultant phasor, stays fixed.

Q: In terms of \( E_R \), \( \alpha \), and \( wt \), what is \( E_1(t) + E_2(t) \)?

A: \( E_1(t) + E_2(t) = E_R \sin (wt + \alpha) \)
- The vector sum of the phasors has vertical component the sum of the waves.

Q: What's $\alpha$?

A: Isosceles $\Rightarrow \alpha = \alpha'
\hspace{0.5cm} x + \phi = 180^\circ
\hspace{0.5cm} x + 2\alpha = 180^\circ \Rightarrow 2\alpha = \phi
\hspace{0.5cm} \alpha = \phi/2$

- The sum of the equal amplitude waves $E_1(t) + E_2(t) = E_0 \sin(\omega t) + E_0 \sin(\omega t + \phi)$ has magnitude $E_R = 2E_0 \cos(\phi/2)
\hspace{0.5cm}$ phase $\alpha = \phi/2$

$E(t) = E_R \sin(\omega t + \alpha) = 2E_0 \cos(\phi/2) \sin(\omega t + \phi/2)$

average intensity of the sum $I_R = \frac{1}{2} (E_0^2) E_R^2 = \left(\frac{E_0}{2}\right) \left(4E_0^2 \cos^2(\phi/2)\right)$
The average intensity of just $E_1(t)$ is

$$I_1 = \frac{1}{2} E_0^2$$

Q: Plot $I_R$ and $I_1$ as a function of the phase difference $\phi$, for $0 < \phi < 2\pi$.

Q: What happens for a phase difference $\phi = \pi$ (one half wiggle)? Draw the phasor diagram.
A: Destructive interference. No intensity.

Q: What happens for a phase difference of zero, or $2\pi$? (No wiggles, two wiggles?) Draw the phasor diagram.
A: Constructive interference. Intensity four times $I_1$.

Constructive interference of two sources gives more than twice the light.
Interference from Two Slits

**DEMO: Laser & Double Slit**

Two Pinholes: Huygens' Principle, \(A + P\)

\[
E_p(t) = \frac{A}{r_1} \sin\left(\frac{2\pi r_1}{\lambda} - 2\pi ft\right)
\]

**Q:** Why \(\frac{A}{r_1}\)?

- Sphere area \(\pi r^2\)

**Q:** Slit \(\frac{A}{r_1^2}\)

\[
\text{Intensity } \frac{1}{r^2} \quad \text{Spherical Waves}
\]

- Cylinder area \(\pi r \sqrt{A}\)

**Q:** Why \(\frac{A}{r_1} + \frac{A}{r_2}\)?

\[
\frac{1}{\lambda} \left[ \sin\left(\frac{2\pi r_1}{\lambda} - 2\pi ft\right) + \sin\left(\frac{2\pi r_2}{\lambda} - 2\pi ft\right) \right]
\]

(Slit \(\approx \frac{1}{r_1^{1/2}}\), messy)

This is a phasor diagram problem!
Calculating $r_2 - r_1$

- Arc of circle (blue) = straight line
- $\theta_1 = \theta_2 = \theta$

Q: What's $r_2 - r_1$?

$\approx d \sin \theta$

$$E_p(t) \approx \frac{A}{r} \left( \sin \left( \frac{2\pi (t - t_0)}{\lambda} - 2\pi ft \right) + \sin \left( \frac{2\pi ft}{\lambda} - 2\pi ft + \phi \right) \right)$$

$$\phi = \frac{2\pi \text{(Path length difference)}}{\lambda} = 2\pi \left( \frac{r_2 - r_1}{\lambda} \right)$$

$$= \frac{2\pi d \sin \theta}{\lambda}$$

- Phase lag $\frac{2\pi d \sin \theta}{\lambda}$; $\frac{dsin\theta}{\lambda}$ extra wiggles

Phasor length

$$= 2 \frac{A}{r} \cos \left( \frac{\phi}{2} \right)$$

$\Rightarrow I_{av} = \frac{4 \frac{A^2}{r^2} \cos^2 \left( \frac{\phi}{2} \right)}{2}$

$I_{av} = I_0 \cos^2 \left( \frac{2\pi d \sin \theta}{\lambda} \right)$, $I_0 = \text{intensity from one slit}$
- Constructive for \( \frac{d \sin \theta}{\lambda} = 0, \pm 1, \ldots \)
- Destructive for \( \frac{d \sin \theta}{\lambda} = \pm \frac{1}{2}, \pm \frac{3}{2} \) (half wiggles)

- For small \( \theta \), \( I_{av} \sim \cos^2 \left( \frac{2\pi d}{\lambda} \theta \right) \)

Double Slit Intensity
(Small \( \theta \))
Multiple Slits / Gratings / Antenna Arrays / Bragg Scattering

Multiple Sources of Waves; Equally Spaced

- Multiple Slits
- Diffraction Grating (CD)
  
[DEMO] Laser & CD
  
6000 lines/cm
  
N = Spot size/groove spacing

- Multiple Radio Transmission Towers
- Atoms Scattering X-rays

Phase Shift Between Waves $\phi = \frac{2\pi}{\lambda} d \sin \Theta$

Amplitude $E(\Theta) = \sum_{n=1}^{N} \cos(2\pi f t + n \phi)$
Phasor Diagram is a Polygon with $N$ sides

- Main Peak $\phi = 0$
  
  \[ I_{av} = N^2 I_{\text{one source}} \]

- First Minimum: Closed
  
  Polygon
  \[ N\phi = \text{Total Angle} \]
  \[ \phi = \frac{2\pi}{N} \]

- Second "Maximum"
  
  Constructive at $N\phi = 3\pi$
  
  \[ \phi = \frac{3\pi}{N} \]
  
  For $N = 6$, $E = \sqrt{2} E_{\text{one source}}$
  
  \[ I_{av} \sim N^2 I_{\text{one source}} \sim \frac{1}{8} I_{av}(\phi=0) \]

  Much smaller than Main Peak $\phi = 0$

Q: What is Second Minimum?
A: $\phi = \frac{3\pi}{N}$

Q: Does $I_{av}$ ever get huge again?
A: $\phi = 2\pi$!
Double Slit

\( N = 2 \)

\( N = 6 \)

\( \phi \)

0

\( \frac{2\pi}{N} \)

2\pi

4\pi

\( \sin \theta \)

0

\( \frac{\lambda}{Nd} \)

Zeroth-Order

Maximun

First-Order

Maximum

First Minimum

Second-Order

Maximum

- Big \( N \) \( \Rightarrow \) Sharper Peaks
  at \( d \sin \theta = m \lambda \)

- Position of peak measures \( \lambda \) (Diffraction grating, spectrometer)

\( \text{With Bigger } N \Rightarrow \text{Better measurement of } \lambda \)

(Easier to read \( \sin \theta = \frac{\lambda}{d} \) if peak sharp)
**Demo: Single-Slit Diffraction**

- Divide slit into $N$ chunks, size $\Delta x = \alpha / N$

- Phase difference $\Delta \phi$ between neighbor chunks:

  \[
  \Delta \phi = 2\pi \frac{\Delta x \sin \theta}{\lambda}
  \]

- $E(\theta) = \sum_{n=1}^{N} (\text{field of chunk } n \text{ at point } P)$

  \[
  = \sum_{n=1}^{N} (\Delta E_0) \sin(2\pi ft + n \Delta \phi)
  \]
Phasor diagram = Polygon of chunk contributions

\[ \Delta \phi = \frac{2\pi \Delta x \sin \theta}{\lambda} \]
gives the biggest intensity?

\[ \Delta \phi = 0 \]
A+ center, \( \theta = \Delta \phi = 0 \)
\[ E_{\text{center}} = N \Delta E_0 \sin(2\pi f t) \]
\[ I_{\text{center}} = \frac{\varepsilon_0}{2} |E_{\text{center}}|^2 = \frac{\varepsilon_0}{2} N^2 (\Delta E_0)^2 \]

\[ \text{Q} \quad \text{At what } \Delta \phi \text{ will } E(\theta) = 0? \]

\[ \Delta \phi = 2\pi \]
Polygon closes when \( N \Delta \phi = 2\pi \)
\[ N \Delta \phi = 2\pi \cdot N \left( \frac{2\pi \Delta x \sin \theta}{\lambda} \right) = 2\pi \]
\[ (N \Delta x) \sin \theta = \lambda \]
\[ a \sin \theta = \lambda \quad \text{First dark spot single slit} \]
\[ d \sin \theta = \lambda \quad \text{First light spot double slit} \]

When path length difference between top & bottom is one wiggle, all phases are added \( \Rightarrow \) complete cancellation!
General Case!

- Chunk contributions form arc of a circle radius $R$, angle $2\alpha$.

**Q:** What is $E(\theta)$, in terms of $R$ and $\alpha$?

**A:** $E(\theta)/2 = R \sin \alpha$

\[ E(\theta) = 2R \sin \alpha \]

Need $R$ and $\alpha$.

\[ 2\alpha = \left[ \text{Angle phasor rotates} \right] = N \Delta \phi \]

\[ \alpha = \frac{N}{2} \Delta \phi = \frac{N}{2} \left( 2\pi \frac{\Delta \phi \sin \theta}{\lambda} \right) = \pi \frac{(N \Delta \phi \sin \theta)}{\lambda} = \pi \alpha \sin \theta \]

\[ \alpha = \pi \alpha \sin \theta \]

Arc length $= R \cdot (\text{angle of arc})$

\[ E_{\text{center}} = N \Delta E_0 = R \cdot 2\alpha \]

\[ R = E_{\text{center}} \left( \frac{1}{2\alpha} \right) \]

\[ E(\theta) = 2R \sin \alpha = 2 \frac{E_{\text{center}} \sin \alpha}{2\alpha} = E_{\text{center}} \sin \frac{(\sin \alpha)}{\alpha} \]

\[ I(\theta) = I_{\text{center}} \sin^2 \frac{\alpha}{\lambda} \]

Computer DEMO Dugan's Dynamic Phasors
**BEATS, A.M. RADIO, AND FOURIER TRANSFORMS**

**DEMO: Beats**

Adding two waves of the same amplitude, different frequencies,

\[ A(t) = A_1 \sin(2\pi f_1 t) + A_2 \sin(2\pi f_2 t) \]

**Envelope** (Smooth curve over wiggles)

Two beats per period of envelope.

When \( f_1 \) and \( f_2 \) are close, get beats.

**Phasor Diagram**

\[ A(t) = A_e(t) \sin(\omega t) \]

\( f_1 \approx f_2 \Rightarrow \) nearly rigid \( \Rightarrow \) \( A_e \) changes slowly

\( \Rightarrow \) envelope
$A(t) = A_1 \sin(2\pi f_1 t) + A_2 \sin(2\pi f_2 t)$

- Looks like a wave, frequency $\bar{f} = \frac{f_1 + f_2}{2}$ pulsing off and on.
- Write $f_1 = \bar{f} - \frac{\Delta f}{2}$, $f_2 = \bar{f} + \frac{\Delta f}{2}$, $\Delta f = f_2 - f_1$

$A(t) = A_1 \sin(2\pi (\bar{f} - \frac{\Delta f}{2}) t) + A_2 \sin(2\pi (\bar{f} + \frac{\Delta f}{2})t)$

$= A_1 \sin(2\pi \bar{f} t - \pi \Delta ft)$
$+ A_2 \sin(2\pi \bar{f} t + \pi \Delta ft)$

Now, $\sin(B + C) = \sin B \cos C + \cos B \sin C$
$\sin(B - C) = \sin B \cos C - \cos B \sin C$

$A(t) = A_1 \left[ \sin(B - C) + \sin(B + C) \right]$

$= A_1 \left[ \sin B \cos C + \cos B \sin C\right.\
\left. + \sin B \cos C - \cos B \sin C \right]$

$= 2A_1 \sin B \cos C = 2A_1 \sin(2\pi \bar{f} t) \cos(\pi \Delta ft)$

$= 2A_1 \cos(\pi \Delta ft) \sin(2\pi \bar{f} t)$

Envelope $A_E(t)$ oscillates at $\Delta f/2$ but two beats per period

Carrier Frequency at $\bar{f} = \frac{f_1 + f_2}{2}$
\[ \Delta f = f_2 - f_1 \]
\[ \bar{f} = \frac{f_2 + f_1}{2} \]

\[ \phi_E(t) = \text{angle of } A \bar{E} = 2\pi f_1 t + \pi \Delta ft \]
\[ = 2\pi f_1 t + \pi (f_2 - f_1) t = \pi f_1 t + \pi f_2 t = 2\pi \bar{f} t \]

\[ A_E(t) = 2A_1 \cos (\pi \Delta ft) \quad \text{Envelope} \]

\[ A(t) = A_E(t) \sin \phi_E(t) \]
\[ = \left[ 2A_1 \cos (\pi \Delta ft) \right] \sin (2\pi \bar{f} t) \quad \text{Envelope} \quad \text{Carrier} \quad \text{Frequency} \]

**WARNING:** Two beats per period.

**This is what AM radio does!**

- **AM = Amplitude Modulation**

\[ \Delta f = \text{frequency of note } A = 440 \text{ Hz} \]
\[ \bar{f} = \text{frequency of radio station } 80 = 800 \text{ kHz} \]

Signal is \[ 2A_1 \cos (\pi \Delta ft) \sin (2\pi \bar{f} t) \] sum of 800,220 Hz and 799,780 Hz
III. QUANTUM MECHANICS

1686 Newton → Gravity, Laws, Thermo, Hydrodynamics

1873 Maxwell ↔ Electricity, Magnetism → Radio...

Universe as Giant Clock: Determinism

1900-1930 Quantum Mechanics

Welcome to the 20th Century!

Today: Wave-particle duality of light

Greeks → Newton: Light made of particles
   Reflection = Bouncing

Haygens → Maxwell: Light is a Wave
   Diffraction, Interference

Quantum Mechanics: Light is Weird
   (Everything else is Weird Too)

\[ \nu = \frac{E}{h} \]

\[ E = hf \]

\[ \Phi \]

\[ \Phi = \text{Work function for metal (energy to leave)} \]

\[ \text{Photoelectric Effect} \]

Light (photo) causes electrons (electric) to pop off metal

Electrons leave metal if their energy > \( \Phi \)
Newton: Light as Particle

- Energy of electron depends on kind of light (color)
  - Maximum energy = \( E_{\text{photon}} - \Phi \)
- Number of electrons proportional to intensity of light (number of light particles)

Maxwell: Light as Wave

- More intensity \( \rightarrow \) bigger \( E \)
  \( \rightarrow \) faster electrons

\[ \text{WRONG!} \]

**DEMO: PHOTOELECTRIC EFFECT**

Light sometimes acts like a particle!
Quantitative Experiment

Maximum Ejected Energy

Maximum Energy

\[ \text{Maximum Energy} = hf - \phi \]

Photon Energy lost leaving metal

\[ w = \text{energy} \]

\[ \text{slope} \frac{\text{intercept}}{h} \]

\[ f \]

\[ \phi \]

\[ \text{intercept} \quad \phi \]

\[ \text{Photon Energy} = hf \]

- \( h = \text{Planck's constant} \)

- \( h = \frac{1}{2\pi} \) also called Planck's Constant or "aitch bar"

\[ E_{\text{photon}} = hf = \hbar w \quad w = 2\pi f = \text{angular frequency} \]

- \( h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s} \)

- \( \hbar \approx 1 \times 10^{-34} \text{ J} \cdot \text{s} \)

Very small! But the eye can see \( \sim 6 \text{ photons} \)
The de Broglie Wavelength

What is the momentum of a photon?

My first prelim:
Flashlight in space kefton
All energy in batteries
drains into light
How fast is flashlight moving?

Momentum of Light

\[ \text{Momentum of Photon} = \frac{\text{Energy}}{\text{P}} \]

\[ = \frac{hf}{c} = \frac{h}{\lambda} \quad (\text{since } f\lambda = c) \]

Wavelength = \lambda = \frac{h}{p}

Louis de Broglie: If light is both wave and particle, why not also matter?

\[ \lambda = \frac{h}{p} \quad \text{for electrons, any particles, ...} \]
Atomic Spectra

Guitars in Zero Gravity

Identical Guitars floating in zero g.

- Collisions
- Pluck one another's strings

Can tell type of guitars
Each string $f_0, 2f_0, 3f_0, ...$

Strings E, C, G (C)

When Atoms Collide, sometimes they emit light!

Important Difference

Atoms only emit light at frequencies corresponding to differences in their internal frequencies

DEMO: Glowing Hg, H, He, & Diffraction Gratings

Mercury Lamp
Neon Light
Helium in Sun

Story of discovery in Sun
The Bohr ring Atom

- Just as hokey as it sounds
- Got the right answer twelve years later, before the right explanation
- Hydrogen Atom: Circular Orbit around Nucleus ("Sun")
- Waves on a String: Only orbits with

\[ n \lambda = 2\pi r \]

# of de Broglie Circumference wavelengths

- de Broglie: \( \lambda = \frac{h}{p} = \frac{h}{mv} \neq \frac{2\pi \hbar}{mv} \)

So

\[ n \lambda = 2\pi r \]

\[ n \left( \frac{2\pi \hbar}{mv} \right) = 2\pi r \]

\[ n \hbar = mv R \]

mvr = \( \vec{p} \times \vec{r} \) = angular momentum

Angular momentum is quantized in units of \( \hbar \)

...later...
• Energy of electron in orbit with radius r

\[ E = \text{Potential Energy} + \text{Kinetic Energy} \]

\[ = -\frac{ke^2}{r} + \frac{1}{2} mv^2 \]

Coulomb's Law: proton + e attracts electron - e

What's \( v^2 \) for circular motion? Coulomb force must balance centripetal acceleration

\[ F = ma = -m \frac{v^2}{r} = -\frac{ke^2}{r^2} \]

\[ \text{circular Force} = -\frac{dV}{dr} \]

\[ \Rightarrow \frac{1}{2}mv^2 = \frac{1}{2} \frac{ke^2}{r} \]

\[ \Rightarrow \text{Energy} = -\frac{ke^2}{r} + \frac{1}{2} \frac{ke^2}{r} = -\frac{1}{2} \frac{ke^2}{r} \]

• Solve for \( r_n \) using \( mvr_n = nh \) and \( \frac{1}{2}mv^2 = \frac{ke^2}{r_n} \)

\[ \left( mr_n^2 \right)^{1/2} mv^2 = \frac{ke^2}{dr_n} \left( mr_n^2 \right) \]

\[ \left( nh \right)^2 = \left( mvr_n \right)^2 = mke^2 r_n \]

\[ r_n = \frac{n^2 \hbar^2}{mke^2} = n^2 a_o \]

\[ a_o = \frac{\hbar^2}{mke^2} = \text{Bohr radius} \]
• Solve for \( E_n \)

\[
E_n = -\frac{k_e e^2}{2r_n} = -\frac{k_e e^2}{2(n^2a_0)} = -\frac{k_e e^2}{2a_0}\left(\frac{1}{n^2}\right)
\]

Bohr's

Standing Wave Solutions
to Hydrogen Atom

\[
E_n = -\left(\frac{k_e e^2}{2a_0}\right)\frac{1}{n^2} \rightarrow \frac{\hbar}{n^2} = -13.6 \text{ eV} \left(\frac{n^2}{n^2}\right)
\]

\( \text{leV} = 1.6 \times 10^{-19} \)

= energy gained
by electron in
one volt
potential drop

Q: Energy of Photon

\[
= E_3 - E_2
= -13.6 \left(\frac{1}{9} - \frac{1}{4}\right) \approx -1.9 \text{ eV}
\approx 3 \times 10^{-19} \text{ J}
\]

E3

E2

Q: Frequency of Photon?

\[
\hbar f = 3 \times 10^{-19} \text{ J}
\]

\[
f = \frac{3 \times 10^{-19}}{6 \times 10^{-34}} \approx 5 \times 10^{14}
\]

\[
\lambda = c/f = 3 \times 10^8/5 \times 10^{14} = 6 \times 10^{-7} \text{ m}
\]

= 600 \text{ nm}
Electron Diffraction, Probability, and the Wave Function

10,000 Volt Electrons

d = 10^{-10} m

Q1: What is the wavelength of an electron with kinetic energy E = 10^4 eV?

Hint: \( m_e = 10^{-30} \) kg \( e = 1.6 \times 10^{-19} \) Coulombs

\[ \frac{1}{2} m v^2 = P^2 / 2m \]

\[ \lambda = h / P \]

\[ P^2 / 2m = E \]

\[ P = \sqrt{2mE} \]

\[ A = e V = (1.6 \times 10^{-19}) (10^4) = 1.6 \times 10^{-15} \]

\[ p = \sqrt{2mE} \]

\[ \lambda = h / p = \frac{h}{\sqrt{2mE}} = 6.6 \times 10^{-34} / \sqrt{2 \times 10^{-30} \times 1.6 \times 10^{-19} \times 10^4} \]

\[ \approx 1.2 \times 10^{-11} \text{ m} \]
Q2: At what angle \( \theta_{\text{min}} \) will the electrons from the two slits destructively interfere?

\[
\theta_{\text{min}} = \frac{1}{2} \lambda
\]

\[
\sin \theta_{\text{min}} = \frac{1}{2} \frac{\lambda}{d} = \frac{1}{2} \frac{1.2 \times 10^{-4} \text{ m}}{10^{-10} \text{ m}} = 0.06
\]

\[\theta_{\text{min}} \approx 3.5^\circ\]

Q3: How many electrons hit \( P_{\min} \) per second, with both slits open?

Q4: If I close one slit, letting no electrons through it, do I get fewer electrons at \( P \)?

Q3: None

Q4: No, blocking one slit yields more electrons at \( P \).

How can this be?

- Electrons are weird
- This happens even when one electron at a time!
- Each electron goes partly through both slits, amplitudes cancel off
- Electron diffraction off surfaces really happens
- Multiple "Antenna" Sources: Atoms on surface

**DEMO: Electron Diffraction**
Q5: Why don't we notice people diffracting? What is \( \lambda \) for a 50 kg physicist jogging at \( v = 2 \text{ m/s} \)?

\[ \lambda = \frac{6.6 \times 10^{-34} \text{ kg m}^2/\text{s}}{(50 \text{ kg})(2 \text{ m/s})} = 6.6 \times 10^{-36} \text{ m} \]

Heavy particles \( \Rightarrow \) Short wavelengths
Slits need to be \( \sim \) wavelength to see diffraction
Light passing through windows \( \Rightarrow \) nearly straight
Physicists passing through doors

Interference effects in light hard to see except for thin slits & light \( \ll \) size of door

Interference effects in sound easy to observe: \( \times \) sound \( \approx \) door size

"That's why it's easy to hear a person even when you can't see me - sound diffracts around edges of doors!"

Interference of physicists very hard to observe, because a physicist \( \lambda \) is so small,
If electrons and light come in lumps, how can they interfere and diffract?

How come more electrons at P_{min} when fewer holes to pass through?

Wave equations $\Rightarrow$ intensity $I(y)$ at screen,

$I(y)$ gives the probability that an electron or photon will hit near $y$.

For photons, electric fields $E(y)$ add:
when $E_L(P_{min}) + E_R(P_{min}) = 0$ no photons.

For electrons, wave functions $\Psi(x)$ add

$\Psi(x) = "psi(x)"$ density

$\Psi(x)^2$ is probability that electron is near $x$

If $\Psi_R(P_{min}) = -\Psi_L(P_{min})$, when both slits are open no electrons will hit at $P_{min}$. 
Schrödinger's Equation

Instead of deriving it (we can’t), we’ll tell it to you.

\[ i\hbar \frac{\partial^2 \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x) \Psi(x,t) \]

- Can’t be derived from Newton’s laws
- Newton’s laws can be derived from it
- Describes particle of mass \( m \) in one dimension \( x \)
- \( V(x) \) = potential energy for particle at \( x \)
- \( \Psi(x,t) \) = wave function

\( i = \sqrt{-1} \)

Complex Numbers Lightning Review

- You plot \( z = a + bi \) in the complex plane
- You write \( z = r e^{i\theta} = r \cos \theta + i r \sin \theta \)

- \( r = |z| = \sqrt{a^2 + b^2} = \sqrt{z z^*} \)
- \( z^* = a - i b \)

\[ e^{i\theta} = \cos \theta + i \sin \theta \]
\[ \frac{d}{d\theta} e^{i\theta} = \left\{ \begin{array}{l} i e^{i\theta} \\ -\sin \theta + i \cos \theta \end{array} \right\} \]
What does $\Psi$ mean?

- Analogous to $E$-field
  - Probability density for photons $\propto I \times \vec{E}^2$
  - Probability density for electrons $\propto |\Psi|^2 = \frac{E_x^2}{\sqrt{(E_x^2 + E_y^2)^2}}$

- $|\Psi(x,t)|^2 \delta x = \text{Probability of finding electron between } x \text{ and } x + \delta x$

- $\int |\Psi(x,t)|^2 dx = 1$ for electrons in bound states (particles in wells)
  - It's gotta be somewhere!
  - **NORMALIZATION**

- Free particles, unbound particles can't be normalized [much math in quantum... not 24(k)]

- $\Psi(x)$ is like a square root of a probability density, except for the i's.

Add $\Psi$'s, then square for probability (add $E$'s, then square for intensity)
Free Particle Schrödinger Equation

Free \Rightarrow V(x) = 0 \Rightarrow \text{No potential energy well}

\[ i\hbar \frac{\partial^2 \psi}{\partial t^2} = -\frac{\hbar^2}{2m} \frac{\partial^4 \psi}{\partial x^4} \]

How shall we solve this partial differential eqn? Guess

\[ \psi(x,t) = \sin \left( \frac{2\pi x}{\lambda} - 2\pi ft \right) \]

But \[ \frac{\partial^4 \psi}{\partial t^2} \sim \cos, \quad \frac{\partial^4 \psi}{\partial x^4} \sim -\sin \]

Also, pesky \[ i = \sqrt{-1} \]

Fix both problems at once! combine \[ \cos \theta + i \sin \theta = e^{i\theta} \]

\[ \psi(x,t) = e^{i(\frac{2\pi x}{\lambda} - 2\pi ft)} = e^{i(kx - \omega t)} \]

Introduce angular frequency \[ \omega = 2\pi f \]

[radians/second]

and wave number \[ k = \frac{2\pi}{\lambda} \]

[radians/meters]

\[ i\hbar \frac{\partial^4 \psi}{\partial t^2} = i\hbar (-i\omega) e^{i(\frac{2\pi x}{\lambda} - \omega t)} = i\hbar \omega e^{i(kx - \omega t)} = \hbar \omega \psi \]

Energy of Electron

\[ (\hbar \omega = hf) \text{ (like photons!)} \]
\[ \frac{\partial^2 \psi}{\partial x^2} = i \frac{\hbar}{m} e^{i(kx - \omega t)} \quad \frac{\partial^2 \psi}{\partial x^2} = i \frac{\hbar}{m} e^{i(kx - \omega t)} \]

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = \left( \pm \frac{\hbar^2}{2m} \right) (\pm \hbar^2) e^{i(kx - \omega t)} \]

\[ \psi(x, t) = e^{i(kx - \omega t)} \text{ satisfies the free particle Schrödinger equation if} \]

\[ \hbar \omega = \frac{\hbar^2}{2m} k^2 \]  

**Dispersion Relation**

(like one-dimensional crystal)

\[ \hbar \omega = hf = \text{energy} \quad \text{for} \quad \text{like photons} \]

\[ \frac{\hbar^2}{2m} k^2 = \left( \frac{2\pi \hbar}{\lambda} \right)^2 = \frac{(h/\lambda)^2}{2m} = \frac{P^2}{2m} \]

De Broglie

\[ \lambda = \frac{h}{p} \]

\[ E = \frac{p^2}{2m} \]

\[ E = \frac{mv^2}{2} \]

Schrödinger’s free equation says total energy = kinetic energy

Schrödinger’s full equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi \]

\[ (\text{Total Energy}) \psi = (\text{Kinetic Energy}) \psi + (\text{Potential Energy}) \psi \]
Standing Waves and Eigenstates: A Particle in a Box

Want standing wave solutions, frequency $f = \frac{\omega}{2\pi}$ for Schrödinger's equation

$$i\hbar \frac{2\Psi(x,t)}{2t} = -\hbar^2 \frac{\partial^2 \Psi}{2m \partial x^2} + V(x) \Psi(x,t)$$

- For waves on string, standing waves looked like $\sin \omega t$ or $\cos \omega t$ times $\sin \left( \frac{n\pi x}{L} \right)$.
- Here we combine $\sin$ and $\cos$ into $e^{-i\omega t}$; the $-i$ will cancel the $i$ on the left.

$$\Psi(x,t) = \psi(x) e^{-i\omega t}$$

- Don't be confused: $\Psi$ = capital $\psi$ = $x$ and $t$
- $\psi$ = lower-case $\psi$ = $x$ only
- Notice $|\Psi(x,t)| = |\psi(x)| \cdot e^{-i\omega t} = |\psi(x)|$: $|\psi(x)|$ is probability density

$$\frac{\partial \Psi}{\partial t} = \psi(x) \frac{\partial e^{-i\omega t}}{\partial t} = -i\omega \psi(x) e^{-i\omega t}$$

$$i\hbar \frac{\partial \Psi}{\partial t} = (i\hbar)(-i\omega) \psi(x) e^{-i\omega t} = i\hbar \psi(x) e^{-i\omega t}$$
Standing wave solutions \( \psi_n \) have definite energy \( E_n \) (like waves on a string have definite frequencies \( f_n \)).

The energies \( E_n \) are often called eigenenergies, and \( \psi_n \) are called eigenstates. Close analogy to eigenvalues \( \lambda_n \) and eigenvectors \( \vec{v}_n \) of a matrix \( \mathbf{M} \) : \( \lambda_n \vec{v}_n = \mathbf{M} \vec{v}_n \).

The \( i \)'s disappeared; \( \psi_n(x) \) can be chosen to be real functions. Serway only discusses the time independent equation; scared of complex stuff?
Particle in a Box

Consider an electron confined to a box of length $L$, by a potential

$$V(x) = \begin{cases} 
0 & 0 < x < L \\
\infty & x < 0, x > L 
\end{cases}$$

This is like having given boundary conditions $\psi(0) = 0$, $\psi(L) = 0$, fixed to zero at walls.

What are the states of definite energy?

$$\mathcal{E}\psi = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) \quad \psi(0) = \psi(L) = 0$$

How do we solve this ordinary differential equation?

**GUESS - or - Notice! Simple Harmonic Oscillator**
Try the wave solution $\psi(x) = A \sin \left( \frac{2\pi x}{L} \right)$

$$E \psi = -\frac{\hbar^2}{2m} \left( \frac{2\pi}{L} \right)^2 A \sin \left( \frac{2\pi x}{L} \right) = \frac{\hbar^2}{2mL^2} \psi(x)$$

$$E = \frac{\hbar^2}{2mL^2}$$

Boundary conditions fix $\lambda_n = \frac{2\pi}{L}$

$$\lambda_n = \frac{2\pi}{L}$$

$$E_n = \frac{\hbar^2}{2m\lambda_n^2} = \frac{\hbar^2}{8mL^2}$$

$$\psi_n = \sqrt{\frac{2}{L}} \sin \left( \frac{2\pi n x}{L} \right)$$

Lowest energy state

$$E_1 = \frac{\hbar^2}{8mL^2}$$

Ground State

- Not at $E=0$ confinement costs energy.
- Probability concentrated away from edge.

First excited state

$$E_2 = \frac{\hbar^2}{2mL^2}$$

Wave Function

Probability Density
Energy Level Diagram
Particle in a Box

Just like Bohr atom:

- Light emitted when atom moves from $E_3$ to $E_1$

$$h\nu = E_3 - E_1$$

Q: What is Frequency $\nu$?

A: $$\nu = \frac{E_3 - E_1}{2\pi h}$$

$$= \frac{9h^2}{8mL^2} - \frac{h^2}{8mL^2} = \frac{h^2}{mL^2 \hbar}$$

also

- Light can be absorbed by kick up electron up in level $E_i$
Quantum Tunneling And The Scanning Tunneling Microscope (STM)

How does the current depend on the distance from tip to surface?

Can electrons with $E$ smaller than the barrier $\phi_0$ penetrate?

Let's solve Schrödinger's equation:
- Ignore $V$: needed for current, but not main point
- Electron with energy $0 \leq E < \phi_0$
In the tip region, $V(x)=0$ and $E>0$  
$\Rightarrow$ standing & traveling wave solutions

$$y^E = A \cos \left( \sqrt{2mE} \frac{x}{\hbar} \right) + B \sin \left( \sqrt{2mE} \frac{x}{\hbar} \right)$$

In the vacuum, $E<\phi_0$: Classically Forbidden  
(Not enough energy to be there)

$$E y = -\frac{k^2}{2m} \frac{d^2y}{dx^2} + \phi_0 y$$

$$\frac{d^2y}{dx^2} = -\frac{2m}{\hbar^2} (\phi_0 - E) y$$

Positive constant $\Rightarrow$ not harmonic oscillator, exponential solutions

$$y^E = Ae^{\sqrt{2mE} \frac{x}{\hbar}} + Be^{-\sqrt{2mE} \frac{x}{\hbar}}$$

Exponential Growth $\Rightarrow$ Tunneling  
From Surface to $\phi_0$ $\Rightarrow$ Unimportant

Exponential Decay $\Rightarrow$ Tunneling  
From Tip to Surface

$k = 10^{-34}$ $m = 10^{-31}$ $1 eV = 1.6 \times 10^{-19}$ $J$

$\phi_0 - E = 44 eV \Rightarrow e^{-x/k} \lambda = \frac{\hbar}{\sqrt{2m(\phi_0 - E)}} = \frac{2.7 \times 10^{-10}}{m} \Rightarrow$ Easy to see atoms $\sim 30 \times 10^{-10}$ $m$
Many-Body Quantum Mechanics

Suppose I have an electron and a neutron in a box of length $L$.

- Neutron is neutral: no electromagnetic interaction.
- Electron is a lepton: no strong interaction.
- Noninteracting particles are easy.

\[ \chi(x_e, x_n) = \text{amplitude electron is at } x_e \text{ and neutron is at } x_n. \]

- Two-particle Schrödinger equation

\[
E \chi(x_e, x_n) = -\frac{\hbar^2}{2m_e} \frac{\partial^2 \chi}{\partial x_e^2} - \frac{\hbar^2}{2m_n} \frac{\partial^2 \chi}{\partial x_n^2} + V(x_e - x_n) \chi.
\]

\[ \text{Electron} \quad \text{Neutron} \quad \text{Noninteraction} \]
\[ \text{Kinetic Energy} \quad \text{Kinetic Energy} \]

How shall we solve this many-particle partial differential equation?

**GUESS.** Since electron doesn't notice the neutron, \[ \chi_e(x_e) = \sqrt{\frac{3}{L}} \sin \left( \frac{n_e \pi x_e}{L} \right) \]

Since neutron doesn't notice \[ \chi_n(x_n) = \sqrt{\frac{3}{L}} \sin \left( \frac{n_n \pi x_n}{L} \right) \]

Since electron.
Q: What is $n_e$, $n_n$ for the picture above?

A: $n_e = 1$, $n_n = 9$.

Q: What is the energy $E_{n_e n_n}$?

A: \[
\frac{\hbar^2}{2m_e} \left( \frac{n_e \pi}{L} \right)^2 + \frac{\hbar^2}{2m_n} \left( \frac{n_n \pi}{L} \right)^2 = \frac{\hbar^2 n_e^2}{8m_e L^2} + \frac{\hbar^2 n_n^2}{8m_n L^2}
\]

For distinguishable, non-interacting particles:
- Eigenstates are products of single-particle eigenstates:
  \[\Psi_{n_e n_n}(x e x n) = \Psi_{n_e}(xe) \Psi_{n_n}(xn)\]
- Energies are sums of single-particle energies:
  \[E_{n_e n_n} = E_{n_e} + E_{n_n}\]

$E^e_2$ - $E^e_1$

Energy Level Diagram:
- Draw two diagrams, label occupied levels.

Q: Which level $E_{n_n}^n$ has larger energy than $E^e_2$? $n_n = 1838 m_e L^2 / \hbar^2$; $n_e = 43$
Many-Body Quantum Mechanics

Two Distinguishable Particles

\[ |\psi(x_e, x_n)|^2 = \text{Probability density at } x_e, n \text{ at } x_n \]

\[ E \psi = -\frac{\hbar^2}{2m_e} \frac{\partial^2 \psi}{\partial x_e^2} - \frac{\hbar^2}{2m_n} \frac{\partial^2 \psi}{\partial x_n^2} \]

\[ \text{Kinetic Energy of Electron} \]
\[ \text{Neutron Kinetic Energy} \]

**Guess:** Since no interaction, electron must be in electron eigenstate & neutron in neutron eigenstate

Try \[ |\psi(x_e, x_n)| = \left( \frac{1}{\sqrt{L}} \right) \sin \left( \frac{\pi n_e x_e}{L} \right) \sin \left( \frac{\pi n_n x_n}{L} \right) \]

\[ E_{n_e n_n} = \frac{\hbar^2 n_e^2}{8m_e L^2} + \frac{\hbar^2 n_n^2}{8m_n L^2} \]

Q1: What is \( n_e, n_n \) for wavefunction above? All \( n_e = 1, n_n = 4 \)

Noninteracting, distinguishable particles

Many-body eigenstate

A) Wavefunction is product of single-particle wave functions eigenstates

B) Energy is sum of single-particle energies
Q21. Show (A) and (B) without taking any derivatives!

Hint! You know

\[ E_{ne}^e \psi_{ne}(xe) \psi_{nn}(xn) = -\frac{k^2}{2m_e} \frac{\partial^2}{\partial x_e^2} \left[ \psi_{ne}(xe) \psi_{nn}(xn) \right] \]

\[ + E_{nn}^n \psi_{ne}(xe) \psi_{nn}(xn) = -\frac{k^2}{2m_n} \frac{\partial^2}{\partial x_n^2} \left[ \psi_{ne}(xe) \psi_{nn}(xn) \right] \]

\[ (E_{ne}^e + E_{nn}^n) \psi(xe, xn) = -\frac{k^2}{2m_e} \frac{\partial^2}{\partial x_e^2} - \frac{k^2}{2m_n} \frac{\partial^2}{\partial x_n^2} \]

Energies add, wave functions multiply, no interactions (makes it work)

Energy Level Diagram: Split into Two

\[ E_3^e \quad E_1^q \quad M_n = 1839 \text{ me} \approx 2000 \text{ me} \]

Q1: How big must \( n_n \) be so \( E_{nn}^n > E_1^e \)?

\[ A1: \quad \frac{h^2}{8m_n^2} n_n^2 \geq \frac{h^2}{8m_e^2} \]

\[ n_n^2 \geq \frac{M_n}{m_e} \]

\[ M_n > \sqrt{1839} \approx 42.9 \]

Protons and neutrons are so heavy, often we treat them as classical, fixed
Two Identical Particles

Suppose I have two non-interacting electrons in a box, one in the $n=6$ state and the other in the $n=8$ state:

$$\Psi_{68}(x_1, x_2) = \frac{2}{L} \sin\left(\frac{6\pi x_1}{L}\right) \sin\left(\frac{8\pi x_2}{L}\right)$$

THIS IS WRONG!

Electrons are identical.

Q: What's the probability that an electron is at $A$ and the other electron is at $B$?

$$A: |\Psi_{68}(A, B)|^2 = 40 + |\Psi_{68}(B, A)|^2$$

The trial wave function above is wrong because these have to be equal to each other:

$$\sin\left(\frac{6\pi A}{L}\right) \sin\left(\frac{8\pi B}{L}\right) \neq \sin\left(\frac{6\pi B}{L}\right) \sin\left(\frac{8\pi A}{L}\right)$$

Two Choices:

$$\Psi_{68}(A, B) = \pm \Psi_{68}(B, A)$$

- Plus Sign $\Rightarrow$ Bosons
- Minus Sign $\Rightarrow$ Fermions
All particles are either fermions or bosons.

A wave function of identical bosons is unchanged if the bosons swap positions.

A wave function of identical fermions gets a minus sign any time two particles swap positions.

Q: Name some particles: Fermion or Boson?

**FERMIIONS**
- electrons
- protons
- neutrons
- quarks
- neutrinos

**BOSONS**
- photons
- Hydrogen
- Helium

Recent Bose Condensation (last month)

Swap electron \((-1)\)

Swap proton \((-1)\)

Boson \(1\)

Helium \((2\text{ electrons} \ 2\text{ protons} \ 1\text{ neutron})\)

Richardson, Lee, Osheroff
Nobel Prize 1996
\[ \psi_{68}(x_1, x_2) = \Psi \left[ \sin \left( \frac{6\pi x_1}{L} \right) \sin \left( \frac{8\pi x_2}{L} \right) - \sin \left( \frac{6\pi x_2}{L} \right) \sin \left( \frac{8\pi x_1}{L} \right) \right] \]

- Plus sign for bosons

- \( \psi_{68}(A, B) = - \psi_{68}(B, A) \) Antisymmetric

- \( \psi_{68}(x_1, x_2) \) solves Schrödinger's equation,

\[ \left[ \text{Each piece has energy } E_{68}^a, E_{68}^b, E_{68}^c \right. \text{ sum of eigenstates with energy } E \text{ of eigenstate of energy } E \]

- For non-interacting identical fermions, wave functions multiply then antisymmetrize still

- Energy is the sum of single-particle energies
Fermions, Bosons, & Level Diagrams for Identical Particles

Last Thursday: Two Electrons in a Box

\[ \psi_{68}(x_1, x_2) \neq \frac{N}{2} \sin\left(-\frac{6\pi x_1}{L}\right) \sin\left(-\frac{8\pi x_2}{L}\right) \]

\[ \psi_{68}(A, B) \psi_{68}(B, A) = \frac{1}{2} \left| \psi_{68}(A, B) \right|^2 \text{ for identical particles} \]

\[ \psi_{68}(A, B) = -\psi_{68}(B, A) \text{ for electrons, fermions} \]

Can we find an eigenstate? Try antisymmetrizing!

\[ \psi_{68}(x_1, x_2) = N \left[ \sin\left(-\frac{6\pi x_1}{L}\right) \sin\left(-\frac{8\pi x_2}{L}\right) - \sin\left(-\frac{8\pi x_1}{L}\right) \sin\left(-\frac{6\pi x_2}{L}\right) \right] \]

Both parts are eigenstates with the same energy \( E_6 + E_8 \Rightarrow \psi_{68} \text{ is an eigenstate.} \)

- The eigenstate for many fermions is the antisymmetrized product of single-particle eigenstates

- The energy is the sum of the single-particle energies
Q: What difference does this make?

A: Two differences.

1) Wave function changes shape
\[ \psi(x, x) = 0 \] for example. Homework

2) Energies are unchanged
EXCEPT: can't put two electrons into the same state!

\[ \psi(x_1, x_2) = N \left[ \sin \left( \frac{6\pi x_1}{2} \right) \sin \left( \frac{6\pi x_2}{2} \right) - \sin \left( \frac{6\pi x_2}{2} \right) \sin \left( \frac{6\pi x_1}{2} \right) \right] = 0 \]

Pauli's Exclusion Principle
No two fermions can occupy precisely the same state.
One more fact needed before we draw electron level diagrams: **spin**.

- **Weird Fact**: All fermions (and most bosons) have an extra quantum number, called spin.
- The spin of an electron can take two values, $\pm \frac{1}{2}$.
- An electron of spin $\frac{1}{2}$ has angular momentum $\hbar/2$, $-\frac{1}{2}$ has angular momentum $-\hbar/2$.
  (Spin means angular momentum is quantized in $\hbar/2$, not $\hbar$.)
- Light has spin too: right-handed polarized light has spin 1 and left-handed has spin -1.
  (Light, because it's massless, doesn't have a spin zero state.)

So, each single-particle eigenstate can have two electrons, one ↑, the other ↓:

![Diagram](image.png)
The 3-D Schrödinger Equation

\[ i \hbar \frac{\partial \psi(x, y, z, t)}{\partial t} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \]

\[ + V(x, y, z) \psi \]

Potential Energy

Guess: \( \psi(x, y, z) = N \sin \left( \frac{2\pi x}{L} \right) \sin \left( \frac{m_1 \pi x}{H} \right) \sin \left( \frac{m_2 \pi z}{W} \right) \)

\[ = N \sin (k_x x) \sin (k_y y) \sin (k_z z) \]

\( k = (k_x, k_y, k_z) = \frac{2\pi}{L} \left( m_1, m_2, m_3 \right) = \text{wave vector} \)

\[ \frac{\partial^2 \psi}{\partial x^2} = -k_x^2 \psi \]

Eigenstates

\[ E \psi(x, y, z) = \frac{\hbar^2}{2m} \left( k_x^2 + k_y^2 + k_z^2 \right) \psi \]

\[ = \frac{\hbar^2}{2m} k^2 \quad \text{Energy depends only on } k^2 \]
The Free Electron Theory of Metals

- Most of the electrons in a metal are tightly bound to the nuclei: core electrons per atom
- One or two electrons in the "outer shell" are relatively free to move through the metal
- In many cases, ignoring the interaction of these free and free electrons with each other and with the nuclei and core electrons is an OK approximation.

Free Electron Theory of Metals

\[ N = \text{N electrons in a } L \times L \times L \text{ Box} \]
\[ L = 1 \text{ cm} \Rightarrow N \approx 10^{23} \]

What's the Ground State?

- Fill smallest energies first, up to Fermi
- Energy \( \propto \hbar^2 \)
- Energy \( E_F \)
- Fill up together sphere in \( \mathbf{k} \) space

Fermi Sphere
Atoms and the Periodic Table

The Hydrogen Atom: Can Solve It!

\[ E_1 = -\frac{\hbar^2}{2m} \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \right) - \frac{k\hbar^2}{r} \]

"Guess" Ground State

\[ \psi_{1s}(r) = N e^{-r/\alpha_0} = N e^{-\sqrt{x^2+y^2+z^2}/\alpha_0} \]

Plug it in! We use chain rule \( \frac{df}{dx} = \frac{df}{dr} \frac{dr}{dx} \),

and \( \frac{dr}{dx} = \frac{d\sqrt{x^2+y^2+z^2}}{dx} = \frac{1}{2\sqrt{x^2+y^2+z^2}} (2x) = \frac{x}{r} \)

\[ \frac{\partial \psi_{1s}}{\partial x} = -\frac{1}{\alpha_0} e^{-r/\alpha_0} \left( x/r \right) \]

Product Rule \( \frac{d(\psi g)}{dx} = \frac{df}{dx} g(x) + \frac{dg}{dx} f(x) \)

\[ \frac{\partial^2 \psi}{\partial x^2} = +\frac{1}{\alpha_0^2} e^{-r/\alpha_0} \left( x/r \right)^2 - \frac{1}{\alpha_0} e^{-r/\alpha_0} \left[ \frac{1}{r} + x \left( \frac{1}{r^2} - \frac{1}{r} \right) \right] \]

\[ = \frac{1}{\alpha_0^2} \frac{x^2}{r^2} e^{-r/\alpha_0} + \frac{1}{\alpha_0} \left( \frac{x^2}{r^2} - \frac{1}{r} \right) e^{-r/\alpha_0} \]
\[ \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \]
\[= \frac{1}{a_o^2} \left( \frac{x^2 + y^2 + z^2}{r^2} \right) e^{-\frac{r}{a_o}} + \frac{1}{a_o} \left( \frac{x^2 + y^2 + z^2}{r^3} - \frac{3}{r} \right) e^{-\frac{r}{a_o}} \]
\[= \left( \frac{1}{a_o^2} - \frac{2}{a_o} \right) \psi \]

\[ E_1 \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi - \frac{ke \varepsilon^2}{r} \psi \]
\[= -\frac{\hbar^2}{2m} \left( \frac{1}{a_o^2} - \frac{2}{a_o} \right) \psi - \frac{ke \varepsilon^2}{r} \psi \]
\[= -\frac{\hbar^2}{2ma_o^2} \psi + \left( \frac{\hbar^2}{2ma_o^2} - \frac{ke \varepsilon^2}{r} \right) \psi \]

\[ E_1 \psi = -\frac{\hbar^2}{2ma_o^2} \psi \]

Solves Schrödinger's Equation

\[ 1s \text{ State ("one ess") } \]

\[ \psi(r) \]

\[ r \psi(r) \]

\[ \text{Nucleus } P^+ \]

\[ \text{Charge distribution} \]

\[ \text{Spherical (s is for spherical)} \]
Excited States of Hydrogen

\[ \Psi_{2s}(x, y, z) = N \left( 2 - \frac{r}{a_0} \right) e^{-r/2a_0} \]

\[ E_{2s} = -\frac{\hbar^2}{2ma_0^2} \left( \frac{1}{4} \right) \]

Three eigenstates

\[ \Psi_{2p}(x, y, z) \propto x e^{-r/2a_0} \]

\[ E_{2p} = -\frac{\hbar^2}{2ma_0^2} \left( \frac{1}{4} \right) y e^{-r/2a_0} \]

\[ E_{2p} = -\frac{\hbar^2}{2ma_0^2} \left( \frac{1}{4} \right) z e^{-r/2a_0} \]

\( p \) states:
- always along \( x, y, z \)
- always three
  - \( l = 1 \) \((-1, 0, +1)\)
Any superposition of eigenstates with energy $E_2$ is still an eigenstate:

\[ \psi_{Pz} = \psi e = 0, m = 0 \]

if angular momentum measured along $z$.

\[ \psi_{Px} = (\psi_{11} + \psi_{1-1}) \]
\[ \psi_{Py} = (\psi_{11} - \psi_{1-1}) \]

Chemical bonds are often described using these superpositions.

Carbon bonds: $4\uparrow$, usually in tetrahedron.

\[ \psi_{2s} + \psi_{Pz} + \psi_{Py} + \psi_{Pz} \]

\[ \Rightarrow \text{Points along } x + y + z \]
\[ n_{3s} = \{ \text{one spherical state, two radial nodes} \} \]

\[ n_{3p} = \{ \text{three } p \text{-states, one radial node each} \} \]

\[ n_{3d} = \begin{align*}
  xy & \cdot e^{-\frac{r}{\lambda a_0}} \\
  yz & \cdot e^{-\frac{r}{\lambda a_0}} \\
  zx & \cdot e^{-\frac{r}{\lambda a_0}} \\
  (x^2 - \frac{1}{3} r^2) & \cdot e^{-\frac{r}{\lambda a_0}} \\
  (y^2 - \frac{1}{3} r^2) & \cdot e^{-\frac{r}{\lambda a_0}} \\
  (z^2 - \frac{1}{3} r^2) & \cdot e^{-\frac{r}{\lambda a_0}}
\end{align*} \]

Five states, \( l = 2 \) \(-2, -1, 0, +1, +2\) \{five\}

Level Diagram for Hydrogen:

<table>
<thead>
<tr>
<th>Energy Level</th>
<th>3s</th>
<th>3p_x</th>
<th>3p_y</th>
<th>3p_z</th>
<th>3d_xy</th>
<th>3d_yz</th>
<th>3d_xz</th>
<th>3d_x^2-3d_y^2</th>
<th>4s</th>
</tr>
</thead>
<tbody>
<tr>
<td>E_2</td>
<td>Li</td>
<td>Be</td>
<td>B</td>
<td>C</td>
<td>N</td>
<td>O</td>
<td>F</td>
<td>Ne</td>
<td></td>
</tr>
<tr>
<td>E_3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ n_{\text{He}} = \text{Noble Gas} \]

Noninteracting Electron Approximation: the Periodic Table? \text{Homework.}

Noble Gases \text{Inert} \text{Filled Shell} 
\text{Big energy gap for next electron}
Contour Surface $sp^3$

$4_{sp^3} = \frac{1}{2} \left( 4_{2s} + 4_{2px} + 4_{2py} + 4_{2pz} \right)$

3 others, form tetrahedron
Bosons, Lasers, & Superfluids

What's a Boson Again?

Distinguishable, Non-Interacting

\[ \Psi_{n_e n_n}(x_e, x_n) = \Psi_{n_e}(x_e) \Psi_{n_n}(x_n) \]

Many-particle eigenstates = Product of single-particle eigenstates

Indistinguishable Fermions; Non-Interacting

\[ \Psi(x_1, x_2) = -\Psi(x_2, x_1) \]

Antisymmetric Wave Function

\[ \Psi_{nm}(x_1, x_2) = N \Psi_n(x_1) \Psi_m(x_2) - \Psi_n(x_2) \Psi_m(x_1) \]

Every many-fermion eigenstates = antisymmetrized product

\[ \Psi_{nn}(x_1, x_2) = 0 \] Pauli Exclusion Principle:

No Two Electrons in Same State

⇒ Periodic Table
⇒ Metals, Semiconductors, ... [Later courses]

Indistinguishable, Non-Interacting Bosons:

\[ \Psi(x_1, x_2) = +\Psi(x_2, x_1) \] Symmetric

\[ \Psi_{nm}(x_1, x_2) = N \left[ \Psi_n(x_1) \Psi_m(x_2) + \Psi_n(x_2) \Psi_m(x_1) \right] \]

Many-boson eigenstates = Symmetrized product
How does a Laser Work?

- Photons are Bosons
- Bosons Like to be in Same State
- Laser = Many Photons in Same State

A laser starts with a bunch of atoms all of whose electrons are in excited states $E_2$.

$E_3 = \ldots$

"Population Inversion" (Normally most in lower states)

$E_2 \rightarrow E_1\ \triangleleft h\nu$

When an atom decays, it emits a photon of energy $E_2 - E_1$, Frequency $E_2 - E_1 / h$, and wavelength $hc / (E_2 - E_1)$.

But it can leave in any direction $\theta$!

Drastic simplification: we'll keep only two angles:

\[ \theta = 0 \quad e^{i k x} \quad (k = \frac{2 \pi}{\lambda} = \frac{2 \pi (E_2 - E_1)}{hc}) \]

\[ \theta = \frac{\pi}{2} \quad e^{i k x} e^{i k y} = \cos(k y) + i \sin(k y) \]

Plane Wave = Traveling Wave

Solution to Schrödinger

Final state of photon = superposition of all outgoing angles

\[ \Phi(x, y) = e^{i k x} + e^{i k y} \]
Now suppose there is already one photon moving in the $x$ direction, and our atom is contributing a second photon. What is the wave function:

$$\Phi(x_1, y_1, x_2, y_2) = e^{-i\kappa x_1}(e^{-i\kappa x_2 + i\kappa y_2})$$

#1 along $x$  

"Equal amplitudes along two directions"

$$+ e^{ikx_1}(e^{i\kappa x_2} + e^{i\kappa y_1})$$

Bosons $\Rightarrow$ must symmetrize

$$= e^{i\kappa x_1}e^{-i\kappa y_2} + e^{i\kappa x_2}e^{-i\kappa y_1} + 2e^{i\kappa x_1}e^{i\kappa x_2}$$

The two is like constructive interference

Probability an electron along $x$ and an electron along $y$ = first two terms = $1^2 + 1^2 = 2$

Probability both electrons along $x$ = $2^2 = 4$

If there are $N$ photons already moving along $x$,

$$\Phi(x_1, \ldots, x_{N-1}, x_N, y_1, \ldots, y_N) = e^{-i\kappa x_1} \ldots e^{-i\kappa x_{N-1}}(e^{-i\kappa x_N + i\kappa y_N})$$

$$+ e^{i\kappa x_1} \ldots e^{i\kappa x_{N-1}}(e^{i\kappa x_N} + e^{i\kappa y_N}) + \ldots$$

$$= Ne^{i\kappa x_1} \ldots e^{i\kappa x_N} + e^{i\kappa x_1} \ldots e^{i\kappa y_1} + \ldots$$

Probability an electron leaves along $x$ = $\sum N^2$

Probability $N$ photons leave along $y$
An atom is $N$ times more likely to emit a photon into a state with $(N-1)$ photons already in it.

This is called **stimulated emission**.

Laser = Light amplification by stimulated emission of radiation.

It helps to have parallel mirrors - re-use the photons and helps aim.

Superfluids work the same way:

Cool $N$ Helium atoms, they all want to be in the ground state:

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But if you're clever, you can get them all in a moving state:

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Once $N$ bosons in the same state, new bosons like to join, not leave & stays flowing forever!