Fermions, Bosons, & Level Diagrams for Identical Particles

Last Thursday: Two Electrons in a Box

\[ \psi_{68}(x_1, x_2) \neq N \sin(\frac{6\pi x_1}{L}) \sin(\frac{8\pi x_2}{L}) \]

\[ |\psi(A, B)|^2 = |\psi(B, A)|^2 \text{ for identical particles} \]

\[ \psi(AB) = -\psi(B, A) \text{ for electrons, fermions} \]

Can we find an eigenstate? Try antisymmetrizing!

\[ \psi_{68}(x_1, x_2) = N \left[ \sin(\frac{6\pi x_1}{L}) \sin(\frac{8\pi x_2}{L}) - \sin(\frac{8\pi x_1}{L}) \sin(\frac{6\pi x_2}{L}) \right] \]

Both parts are eigenstates with the same energy \( E_6 + E_8 \Rightarrow \psi_{68} \) is an eigenstate.

- The eigenstate for many fermions is the antisymmetrized product of single-particle eigenstates

- The energy is the sum of the single-particle energies
Q: What difference does this make?

A: Two differences.

1. Wave function changes shape
   \[ \psi(x,x) = 0 \text{ for example} \]  Homework

2. Energies are unchanged
   \text{EXCEPT: can't put two electrons into the same state!}

\[ \psi(x_1, x_2) = N \left[ \sin \left( \frac{6\pi x_1}{L} \right) \sin \left( \frac{6\pi x_2}{L} \right) \right] = 0 \]

\[ -\sin \left( \frac{6\pi x_2}{L} \right) \sin \left( \frac{6\pi x_1}{L} \right) \]

\text{Pauli Exclusion Principle}
No two fermions can occupy precisely the same state.
One more fact needed before we draw electron level diagrams: SPIN.

- **Weird Fact:** All fermions (and most bosons) have an extra quantum number, called spin.
- The spin of an electron can take two values, $\pm \frac{1}{2}$.
- An electron of spin $\frac{1}{2}$ has angular momentum $\frac{\hbar}{2}$; $-\frac{1}{2}$ has angular momentum $-\frac{\hbar}{2}$.
  (Spin means angular momentum is quantized in $\frac{\hbar}{2}$, not $\hbar$.)
- Light has spin too; i.e., right-handed polarized light has spin $+1$ and left-handed has spin $-1$.
  (Light, because it's massless, doesn't have a spin zero state.)

So, each single-particle eigenstate can have two electrons, one $\uparrow$, the other $\downarrow$: 

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<table>
<thead>
<tr>
<th>State</th>
<th>E₁</th>
<th>E₂</th>
<th>E₃</th>
<th>E₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ground State</td>
<td>2 electrons</td>
<td>Excited States</td>
<td>2 electrons</td>
<td>Ground State</td>
</tr>
</tbody>
</table>
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![Diagram of electron levels with states labeled](image-url)
The 3-D Schrödinger Equation

\[ i\hbar \frac{\partial \psi(x,y,z,t)}{\partial t} = \frac{-\hbar^2}{2m} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + V(x,y,z) \psi \]

Potential Energy

Particle in a 3-D Box

Guess: \( \psi(x,y,z) = N \sin \left( \frac{2\pi x}{L} \right) \sin \left( \frac{m\pi y}{H} \right) \sin \left( \frac{n\pi z}{W} \right) \)

\( k = (k_x, k_y, k_z) = \frac{\pi}{L} \left( \frac{x}{L}, \frac{m}{H}, \frac{n}{W} \right) \)

\( \frac{\partial^2 \psi}{\partial x^2} = -k_x^2 \psi \)

Eigenstates

\( E \psi(x,y,z) = \frac{\hbar^2}{2m} \left( k_x^2 + k_y^2 + k_z^2 \right) \psi \)

\( = \frac{k^2}{2m} \psi \quad \text{Energy depends only on } k^2 \)
The Free Electron Theory of Metals

- Most of the electrons in a metal are tightly bound to the nuclei: core electrons per atom.
- One or two electrons in the "outer shell" are relatively free to move through the metal.
- In many cases, ignoring the interaction of these free and free electrons with each other and with the nuclei and core electrons is an OK approximation.

Free Electron Theory of Metals

- $N$ electrons in a $L \times L \times L$ Box
- $L = 1 \text{ cm} \Rightarrow N = 10^{23}$

What's the ground state?

- Fill smallest energies first, up to Fermi.
- Energy $\propto \hbar^2 k^2$
- Energy $E_F$
- Fill up sphere in $k$ space

Fermi Sphere