Schrödinger’s Equation

Instead of deriving it (we can't), we'll tell it to you.

\[ i \hbar \frac{\partial^2 \psi(x,t)}{\partial t^2} = -\frac{\hbar^2}{2m} \frac{\partial^4 \psi(x,t)}{\partial x^4} + V(x) \psi(x,t) \]

- Can't be derived from Newton's laws
- Newton's laws can be derived from it
- Describes particle of mass $m$ in one dimension $x$
- $V(x) =$ potential energy for particle at $x$
- $\psi(x,t) =$ wave function

- $i \equiv \sqrt{-1}$ Complex Numbers Lightning Review

\[ \text{in complex plane} \]

\[ z = a + bi \]

You plot \( z \) in polar coordinates \( z = re^{i\theta} = r \cos \theta + ir \sin \theta \)

\[ r = |z| = \sqrt{a^2 + b^2} = \sqrt{z \overline{z}} \]

\[ z^* = a - ib \]

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

\[ \frac{d}{d\theta} e^{i\theta} = \begin{bmatrix} i & e^{i\theta} \\ -\sin \theta + i \cos \theta \end{bmatrix} \]
What does $\psi$ mean?

- Analogous to $\mathbf{E}$-field
  - Probability density for photons $\propto I \times \mathbf{E}^2$
  - Probability density for electrons $\propto |\psi|^2 = \frac{\mathbf{p} \psi}{\sqrt{(\mathbf{p} \cdot \mathbf{p}) + m^2 \gamma^2}}$

- $|\psi(x,t)\rangle^2 \delta x = \text{Probability of finding electron between } x \text{ and } x + \delta x$

- $\int |\psi(x,t)\rangle^2 \text{ dx} = 1 \text{ for electrons in bound states}$
  - It's gotta be somewhere!
  - **NORMALIZATION**

- Free particles, unbound particles can't be normalized [much math in quantum... not 247]

- $\psi(x)$ is like a square root of a probability density, except for the $i$'s.

Add $\psi$'s, then square for probability
(add $E$'s, then square for intensity)
Free Particle Schrödinger Equation

Free $\Rightarrow V(x) = 0 \Rightarrow$ No potential energy well

$$i\hbar \frac{\partial^2 \psi}{\partial t^2} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}$$

How shall we solve this partial differential eqn?

**GUESS**

$$\psi(x,t) = \sin \left( \frac{2\pi x}{\lambda} - 2\pi ft \right)$$

But $\frac{\partial^2}{\partial t^2} \sim \cos$, $\frac{\partial^2}{\partial x^2} \sim -\sin$

Also, pesky $i = \sqrt{-1}$.

Fix both problems at once: combine $\cos \theta + i \sin \theta = e^{i\theta}$

$$\psi(x,t) = e^{i \left( \frac{2\pi x}{\lambda} - 2\pi ft \right)} = e^{i(kx-\omega t)}$$

Introduce angular frequency $\omega = 2\pi f$ [radians/second]

and wave number $k = 2\pi / \lambda$ [meters]

$$i\hbar \frac{\partial^2 \psi}{\partial t^2} = i\hbar (-i\omega) e^{i(kx-\omega t)} = \hbar \omega e^{i(kx-\omega t)}$$

Energy of Electron

$$\hbar \omega = hf \text{ (like photons!)}$$
\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial t^2} = \frac{i\hbar}{m} \partial_t \psi - \frac{\hbar^2}{2m} \partial^2 \psi
\]

\[
\psi(x,t) = e^{ikx-\omega t}
\]

satisfies the free particle Schrödinger equation if

\[
\hbar \omega = \frac{\hbar^2}{2m} k^2
\]

\[\text{Dispersion Relation} \] (like one-dimensional crystal)

\[\hbar \omega = \hbar \frac{p}{m} = \text{Energy for} \text{like photons} \]

\[
\frac{\hbar^2}{2m} k^2 = \frac{(2\pi \hbar)^2}{\lambda^2} = \frac{(\hbar/\lambda)^2}{2m} = P^2/2m
\]

Schrödinger's free equation says total energy = kinetic energy

Schrödinger's full equation

\[
\frac{i\hbar}{m} \frac{\partial^2 \psi}{\partial t^2} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi
\]

\[\text{(Total Energy)} \psi = \text{Kinetic Energy} \psi + \text{(Potential Energy)} \psi \]