Discretizing the Wave Equation

How can we solve the wave equation

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

numerically? Can't store the curve $y(x)$ [$\infty$ # points].

**Discretize Space**

Need to sample $y(x,t)$ at discrete set of positions $x_n = n \Delta x$.

Approximate First Derivative

$$\frac{\partial^2 y}{\partial x^2}(x_n) = \frac{\frac{\partial y(x_n + \frac{\Delta x}{2})}{\partial x} - \frac{\partial y(x_n - \frac{\Delta x}{2})}{\partial x}}{\Delta x}$$

Approximate Second Derivative

$$\frac{\partial^2 y}{\partial x^2}(x_n) = \frac{\frac{\partial^2 y(x_n + \Delta x)}{\partial x^2} - \frac{\partial^2 y(x_n - \Delta x)}{\partial x^2}}{\Delta x} = \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x^2}$$
Discrete Approximation for Wave Equation

$$\frac{d^2 Y_n}{dt^2} = \frac{v^2}{\delta x^2} \frac{Y_{n+1} + Y_{n-1} - 2Y_n}{\delta x^2}$$

Aside! Sound in Crystals

Chain of Balls and Springs

Atoms and Chemical Bonds

One-dimensional Crystal, Longitudinal wave

$$\delta x = \text{Spring length} \quad X_n = n\delta x = \text{Undefomed Position}$$

$$X_{n+1} + U_n = \text{Actual Position}$$

Bond B is stretched: length is $\delta x + u_{n+1} - u_n$

Force on atom $K(u_{n+1} - u_n)$ to right

Bond A is squeezed: length is $\delta x + u_n - u_{n-1}$

Force on atom $-K(u_{n-1} - u_n)$ to right,

$$m a = m \frac{d^2 u_n}{dt^2} = K(u_{n+1} - u_n) - K(u_{n-1} - u_n)$$

$$\frac{d^2 y}{dt^2} = \frac{K}{m} \left( u_{n+1} + u_{n-1} - 2u_n \right)$$

Exactly the same equation: \( \frac{v^2}{\delta x^2} \Leftrightarrow \frac{K}{m} \)

Real chain of atoms \(\Leftrightarrow\) Approximate wave equation
Discretize Time

\[ y_n(t) \] must be discretized too: sample at \( n\delta t \).

\[
\frac{\partial^2 y_n}{\partial t^2} \approx \frac{y_n(t+\delta t) + y_n(t-\delta t) - 2y_n(t)}{(\delta t)^2} \approx \frac{v^2}{(\delta x)^2} y_{n+1} - y_n - 2y_n
\]

Solve for future from past & present.

\[
y_n(t+\delta t) = 2y_n(t) - y_n(t-\delta t) + \frac{v^2}{(\delta x)^2} \left[ y_{n+1}(t) - 2y_n(t) + y_n(t) \right]
\]

Up to you to choose \( \delta t, \delta x \), not a velocity.
Initial Conditions

First Time Step:

\[ y_n(st) = 2y_n(0) + y_n(-st) + \frac{v^2 (\frac{st}{L})^2}{2} (y_n(0) - y_n(-0)) \]

Need

\[ y_0(0), y_1(0), y_2(0), \ldots, y_N(0) \]

and

\[ y_0(-st), y_2(-st), \ldots, y_N(-st) \]

to get started. If we begin with a flat string at rest

\[ y_n(-st) = 0, \]

\[ y_n(0) = 0. \]

Boundary Conditions

Equation of Motion for \( y_0, y_N \):

Forced Boundary Condition: \[ y_0(t) = Y(t) \]

\( Y(t) \) = height of human hand at time \( t \).

Fixed Boundary Condition: \[ y_N(t) = 0 \]

Free Boundary Condition (Best Version) at \( N \):

Zero Slope \( \Rightarrow \) Symmetric

\[ y_{n+1} = y_{n-1}, \quad \frac{d^2 y_n}{dx^2} = \frac{Y_n x - y_n - 2x_n}{(x_L)^2} = -2x_n - 2y_n \]