BOUNDARY CONDITIONS AND ENERGY CONSERVATION

What is the equation of motion for the end of the string? Depends on Boundary Condition.

Fixed Boundary Condition
\[ \gamma(0,t) = 0 \]

\( x = 0 \) (Guitars, Springs at Wall)

Q: Does energy flow into or out of a fixed boundary?

A: Energy flow = Power = \(-T \frac{\partial^2 y}{\partial x \partial t}\)

But \( \frac{\partial^2 y}{\partial t} = 0 \) at a fixed boundary

\[ \Rightarrow \text{Energy is conserved.} \]

How to Solve Wave Equation for Fixed Boundary Conditions

DEMO: Pulse on spring inverts at wall. Use SUPERPOSITION.
**SUPERPOSITION**

Suppose $y_1(t)$ and $y_2(t)$ satisfy the wave equation. Then $y_1(t) + y_2(t)$ also solves the equation.

**Proof:**

\[
\frac{\partial^2 y_1}{\partial t^2} = v^2 \frac{\partial^2 y_1}{\partial x^2} \quad \frac{\partial^2 y_2}{\partial t^2} = v^2 \frac{\partial^2 y_2}{\partial x^2}
\]

Use wave eqn

\[
\frac{\partial^2 (y_1 + y_2)}{\partial t^2} = \frac{\partial^2 y_1}{\partial t^2} + \frac{\partial^2 y_2}{\partial t^2} = v^2 \frac{\partial^2 y_1}{\partial x^2} + v^2 \frac{\partial^2 y_2}{\partial x^2}
\]

\[
= v^2 \left( \frac{\partial^2 y_1}{\partial x^2} + \frac{\partial^2 y_2}{\partial x^2} \right) = v^2 \left( \frac{\partial^2 (y_1 + y_2)}{\partial x^2} \right)
\]

$y_1 + y_2$ is solution.

\[t=0\]

*Fictional Pulse*

*Other Side of Wall*

*Flipped $x=0$*

\[t=0\]

*Pulse starts moving left $f(x+vt)$*

\[t=t_f\]

*Pulse leaves moving right $f(-(x-vt))$*

*Flipped, inverted*

*Inverted $x=0$*
Guess Solution \( f(x+vt) - f(-(x-vt)) \)

1. Solves wave equation?
   - \( f(x+vt) \) solves wave equation \( \frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} \) \{traveling wave\}
   - \( f(-(x-vt)) \) solves wave equation \( \frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} \) \{shape \( f(-x) \)\}

   \( \Rightarrow \) [Superposition] \( f(x+vt) - f(-(x-vt)) \) solves wave equation

2. Satisfies initial condition?
   \( f(x+vt) = 0 \) for \( x < 0 \), \( t=0 \)
   \( \Rightarrow f(-(x-vt)) = 0 \) for \( x > 0 \) at \( t=0 \)
   \( \Rightarrow \) at \( t=0 \), "Fictional pulse" not on string yet.

3. Satisfies boundary condition?
   \( y(0,t) = f(x+vt) + f(-(x-vt)) \) at \( x = 0 \)
   \( = f(vt) - f(-(-vt)) \)
   \( = f(vt) - f(vt) = 0 \)
What other boundary condition conserves energy?

\[
\text{Power} = -\tau \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}
\]

\[
\frac{\partial y}{\partial t} = 0 \Rightarrow \text{Power} = 0 \quad \text{Fixed Boundary Condition}
\]

\[
\frac{\partial y}{\partial x} = 0 \Rightarrow \text{Power} = 0 \quad (?)
\]

\[\tau \frac{\partial y}{\partial x} = \text{Vertical component of force on string}\]

\[\tau \frac{\partial y}{\partial x} = F_y \quad \text{ZERO FORCE} \iff \frac{\partial y}{\partial x} = 0 \Rightarrow \text{FREE BOUNDARY}\]

Examples of Sound, Transmission Lines, ...

Superposition

\[f(x+vt) + f(-(x-vt)) \quad \text{not inverted, flipped right}\]

DEMO: TORSIONAL WAVE MACHINE