TRAVELING WAVES

Another Family of Solutions to the Wave Equation

Clues: Voice
- Travels undistorted from Mouth to Ear
  [Not quite: nearby bolt crack!]
  [Far-away lightning rumble...]
- Pulse on Spring
  - Too Fast  • Longitudinal Better
  - A bit crummy
- Any shape
- Pulse on Torsional Wave
  - A bit crummy  • Backward!
  - Any shape
  - Same speed, any shape \textbf{call speed }C.
    - Why? \quad \textbf{new shape} \rightarrow \textbf{same velocity}

How to write a solution \( y(x,t) \) which can have any shape at all?

\[ y(x,t) = f(x - ct) \]
Travelling Waves Solve the Wave Equation

Does $f(x-ct)$ solve the wave equation

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

We need a notation for derivatives of $f$ with respect to its argument. Let the slope of $f$ be $f'$, and the second derivative be $f''$ [Newton's notation]. (So, if $x = x-ct$, then $f' = \frac{df}{dt}$ and $f'' = \frac{d^2f}{dx^2}$).

$$\frac{\partial y}{\partial t} = \frac{d}{dt} [f(x-ct)] = f'(x-ct) \frac{d}{dt} (x-ct)$$

$$= f'(x-ct) [-v]$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{d}{dt} [-vf'(x-ct)] = -v \frac{d}{dt} f'(x-ct)$$

$$= -v f''(x-ct) \frac{d}{dt} (x-ct)$$

$$= v^2 f''(x-ct)$$

$$\frac{\partial y}{\partial x} = \frac{d}{dx} [f(x-ct)] = f'(x-ct) \frac{d}{dx} (x-ct) = f'(x-ct)$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{d}{dx} f'(x-ct) = f''(x-ct)$$
Does \( y(x,t) = f(x-ct) \) satisfy the wave equation
\[
\frac{\partial^2 y}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial x^2}
\]

\[
\frac{\partial y}{\partial t} = -c f'(x-ct) \quad \frac{\partial y}{\partial x} = f'(x-ct)
\]

\[
\frac{\partial^2 y}{\partial t^2} = +c^2 f''(x-ct) \quad \frac{\partial^2 y}{\partial x^2} = f''(x-ct)
\]

**Traveling Wave** \( y(x,t) = f(x-ut) \) satisfies wave equation for any shape \( f \).

\( u = \) pulse velocity = velocity in \( x \) direction of shape.

**Questions #1**: Are these traveling waves?
If so, what is \( u \)?

\[
y(x,t) = \frac{2}{(x-4t)^2+1}
\]

\[
y(x,t) = (x^2+t^2)
\]

\[
y(x,t) = \cos(6x+2t)
\]

\[
y(x,t) = e^{-\log^2[(7x+8t)^2+37]}
\]
Question #2: Pulse with funny shape at $t=0$ travels to right velocity $v = \sqrt{\frac{E}{m}}$.

Draw $y(0, t)$, the height of the chunk at $x=0$.

Question #3: Is this a traveling wave?

"3D" Plots of $y(x, t)$ A standing wave.

Notice shape flips over!
Suppose \( y_1(x,t) \) and \( y_2(x,t) \) solve the wave equation.

So will \( y(x,t) = Ay_1(x,t) + By_2(x,t) \).

\[
\begin{align*}
\frac{\partial^2 y}{\partial t^2} &= \frac{1}{v^2} \frac{\partial^2 y_1}{\partial x^2} \\
\frac{\partial^2 y_2}{\partial t^2} &= \frac{1}{v^2} \frac{\partial^2 y_2}{\partial x^2}
\end{align*}
\]

\[
\frac{\partial y}{\partial t} = \frac{1}{2t} \left[ A \frac{\partial y_1}{\partial t} + B \frac{\partial y_2}{\partial t} \right] = A \frac{\partial}{\partial t} \left( y_1(x,t) \right) + B \frac{\partial}{\partial t} \left( y_2(x,t) \right)
\]

\[
\frac{\partial^2 y}{\partial t^2} = \frac{1}{2t} \left[ A \frac{\partial^2 y_1}{\partial t^2} + B \frac{\partial^2 y_2}{\partial t^2} \right] = A \frac{\partial^2}{\partial t^2} \left( y_1(x,t) \right) + B \frac{\partial^2}{\partial t^2} \left( y_2(x,t) \right)
\]

\[
= A \frac{\partial^2 y_1}{\partial t^2} + B \frac{\partial^2 y_2}{\partial t^2} = A \frac{v^2}{\partial x^2} + B \frac{v^2}{\partial x^2}
\]

use wave equation

\[
= \left[ \text{same things backwards} \right] v^2 \frac{\partial^2 (A y_1)}{\partial x^2} + \frac{\partial^2 (B y_2)}{\partial x^2}
\]

\[
= v^2 \frac{\partial^2}{\partial x^2} \left( A y_1(x,t) + B y_2(x,t) \right) = v^2 \frac{\partial^2 y}{\partial x^2}
\]

\[
\frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2}, y \text{ are linear } \Rightarrow \text{ solutions add}_4
\]
Examples of Superposition:

1. \[ A \sin \left( \frac{2\pi x}{L} \right) \sin \left( 2\pi ft \right) \]
   - Standing wave amplitude
   - Big waves \( \propto \) small waves

2. \[ \sin (x-vt) = \sin(x)\cos(vt) - \cos(x)\sin(vt) \]
   - Angle addition formula
   \[ \sin(A+B) = \sin A \cos B + \cos A \sin B \]
   \[ \sin(A-B) = \sin A \cos B - \cos A \sin B \]
   - Traveling sine wave is superposition of two standing waves.

3. \[ \sin \left( \frac{2\pi x}{L} \right) \cos \left( 2\pi ft \right) \]
   - \[ = \sin \left( \frac{2\pi x + 2\pi ft}{L} \right) + \sin \left( \frac{2\pi x - 2\pi ft}{L} \right) \]
   - Standing wave is superposition of two traveling sine waves.
   - velocity \( v = -\frac{L}{T} \)
   - \( v = \frac{L}{T} \)

4. Draw \( y(x, \frac{N}{4}T) \)

5. Harmonics & Overtones
Is this a traveling wave?
A standing wave?
Is it moving right? Left?
Notice that the shape $y(x)$ flips over as $y(t)!$
Is this a traveling wave?
A standing wave?
What kind of boundary condition does it have?
What kind of wave is this? We call this a "plane wave"