Physics 214 Spring 99—Problem Set 13—Solutions
Handout May 6, 1999

1. Information on the Final Exam

Final Exam:
The final exam will be given on Tuesday, May 18 from 9:00 to 11:30 am, in Barton Hall. It will be written as a 90 minute exam, but all students will have the full period from 9:00 to 11:30 am to complete it. The exam will cover the material in lectures 19-28, problem sets 10-13 (including optional problems), computer lab 3 (schrదhr) and experimental lab 6. The exam will be closed-book, but with a formula sheet provided. CALCULATORS WILL NOT BE NEEDED OR ALLOWED.

2. Finite Step Potential
You are given a 1-dimensional potential with $U(x) = 0$ for $x \leq 0$ and $U(x) = U_0$ for $x > 0$ where $U_0$ is a positive constant. See the figure below.

![U(x) vs x](image)

Ignoring the normalization condition, sketch the real part of a solution $\psi(x)$, vs $x$, to the Schrödinger equation for a particle with energy $E$ where:

a) $E < U_0$.

**SOLUTION:**

The time independent Schrödinger equation in 1 dimension is given by

$$\frac{d^2 \psi}{dx^2} = -\frac{2m}{\hbar^2} (E - U) \psi$$

If we define

$$\frac{2m}{\hbar^2} (E - U) = \begin{cases} k^2 & E - U \geq 0 \\ -q^2 & E - U < 0 \end{cases}$$

then the Schrödinger equation becomes

$$\frac{d^2 \psi}{dx^2} = \begin{cases} -k^2 \psi & E - U \geq 0 \\ q^2 \psi & E - U < 0 \end{cases}$$

Hence the type of solution depends on the sign of $E - U$. We will have oscillatory solutions for $E - U \geq 0$, and exponential solutions for $E - U < 0$. Also we should note that both $\psi$ and $\frac{d\psi}{dx}$ are continuous, finite and single valued.

For case (a), $E < U_0$, $E - U$ is positive for $x \leq 0$ but negative for $x > 0$. Thus we have an oscillatory solution in the region $x \leq 0$. The solution in the right hand region, for $x > 0$, is a decaying exponential, which matches the value and slope of the oscillating wave at $x = 0$. 
b) $E = 2U_0$.

**SOLUTION:** Here $E - U$ is positive everywhere and more specifically

$$E - U = \begin{cases} 
2U_0 & x \leq 0 \\
U_0 & x > 0
\end{cases}$$

Hence the wavenumber $k$ decreases by a factor $\sqrt{2}$ from the left hand region to the right hand region. Thus we should see the wavelength of the wave function increase. This is seen in the following diagram.

3. Particle in a Potential Well (Qualitative)

Consider an electron of mass $m$ trapped in the following potential well, for which $V$ goes to infinity for $x \leq 0$.

<table>
<thead>
<tr>
<th>V(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>L</td>
</tr>
<tr>
<td>x</td>
</tr>
</tbody>
</table>

**SOLUTION:**

Here again $E - U$ is positive but since $E \gg U_0$, then the difference $E - U$ is hardly changed from the left hand region to the right hand region. Thus the wavenumber $k$ is unchanged.
a) Sketch the wave functions $\psi_1$ and $\psi_2$ for the two lowest bound standing wave states with the energies $E_1$ and $E_2$. Do not worry about the normalization condition in making your sketch.

**SOLUTION:** The wave functions have a sinusoidal form where $E - V(x)$ is positive and exponential where it is negative. The wave function must go to zero far from the well and be continuous, with continuous derivative, at $x = L$. The lowest state has zero interior nodes; the next state has one interior node.

![Wave Functions Graph]

b) Does the lowest energy (“ground”) state, $E_1$, have a lower or higher energy than the lowest energy state of an electron in a box, with infinite potential energy on both sides, of the same width $L$? Why?

**SOLUTION:** Each state has a lower energy in this potential than in a box. The wavelength is longer, as can be seen from the sketch in part a), so the momentum, and therefore the energy, is lower.

c) Sketch the real part of the wave function if the electron has an energy of $4V_0/3$. Do not worry about the normalization condition in making your sketch.

**SOLUTION:** Now $E - V(x)$ is positive everywhere, so the wave function is always oscillatory, and has the form of a traveling wave. The wavelength is proportional to $(E - V(x))^{-1/2}$. For $x < L$, $(E - V(x))^{-1/2} = \frac{\sqrt{2}}{2}$, for $x \geq L$, $(E - V(x))^{-1/2} = \sqrt{3/V_0}$. So the wavelength is longer by a factor of two for $x \geq L$ than for $x \leq L$.

![Real Part Graph]

4. **Electron in a Box**

An electron of mass $m$ is confined in a potential which is infinite for $x < 0$, and for $x > L$. The potential varies linearly from 0 to $V_0$ as $x$ varies from 0 to $L$ (see figure below).
a) In the range $0 \geq x \geq L$, give an equation for the electron's kinetic energy $K(x)$, in terms of $E$, $V_0$, $x$, and $L$.

SOLUTION: Since $V(x)$ varies linearly from $0$ to $V_0$ as $x$ varies from $0$ to $L$, we have $V(x) = V_0 \frac{x}{L}$. Using $E = K(x) + V(x)$, we have

$$K(x) = E - V(x) = E - V_0 \frac{x}{L}$$

b) In the same range, give an equation for the deBroglie wavelength of the electron, $\lambda(x)$, in terms of $E$, $V_0$, $x$, $m$, $h$, and $L$.

SOLUTION: The deBroglie wavelength is given by $\lambda = \frac{h}{p}$. Using $p = \sqrt{2mK}$, we have

$$\lambda(x) = \frac{h}{p(x)} = \frac{h}{\sqrt{2mK(x)}} = \frac{h}{\sqrt{2m(E - V_0 \frac{x}{L})}}$$

c) Let the total energy of the electron $E$ be $E \geq V_0$, and let the deBroglie wavelength of the electron $\lambda(0) = \frac{L}{4}$. Sketch a graph of a possible standing wave solution for the electron’s wavefunction in this potential, in the range $0 \geq x \geq L$.

SOLUTION: Since $E \geq V_0$, the wavefunction will be sinusoidal throughout the potential well, with a wavelength equal to $\lambda(x)$. At $x = 0$, the wavelength is $\frac{L}{2}$, corresponding to 2 oscillations in the well; but as $x$ increases, the kinetic energy decreases and the wavelength increases. The wavefunction must also go to zero at $x = 0$ and $x = L$, since the potential is infinite there. Hence the wavefunction looks like:

d) Let the total energy of the electron $E$ be $E < V_0$. Sketch a graph of the standing wave solution for the electron's wavefunction corresponding to the lowest energy state in this potential, in the range $0 \geq x \geq L$.

SOLUTION: Since $E < V_0$, the wavefunction will be sinusoidal through only a portion of the potential well, where $E > V(x)$. For the region
of the well where \( E < V(x) \), the wavefunction will exhibit exponential decay. For the lowest energy state, there will be no interior nodes. The wavefunction must also go to zero at \( x = 0 \) and \( x = L \), since the potential is infinite there. Hence the wavefunction will look like:

\[
\psi(x) = e^{-Kx}
\]

5. Electron Tunnelling
Consider a barrier of height \( V_b = 6 \text{ eV} \), and of thickness \( L = 700 \text{ pm} \). Calculate the energy of an incident electron such that its probability of transmission is \( 1 \) in \( 1000 \).

**SOLUTION:**
The transmission probability is given by

\[
T \approx e^{-2KL}
\]

where

\[
K = \frac{\sqrt{2m(V_b - E)}}{\hbar} = \frac{\sqrt{2mc^2(V_b - E)}}{\hbar c}
\]

In this problem, we know \( T \) and we want to find \( E \). So

\[
\ln T = -2KL = -2\frac{\sqrt{2mc^2(V_b - E)}}{\hbar c}L
\]

Solving for \( E \) gives

\[
E = V_b - \left( \frac{\hbar^2 \ln T}{2mc^2} \right)
\]

We can convert \( \hbar c \) from J\( \cdot \)m to eV\( \cdot \)pm using \( 1 \text{ eV} = 1.6 \times 10^{-19} \text{ J} \) and \( 1 \text{ pm} = 10^{-12} \text{ m} \):

\[
\hbar c = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{m} \times 1 \text{ eV}}{1.6 \times 10^{-19} \text{ J} \times 10^{-12} \text{ m} = 197,500 \text{ eV} \cdot \text{pm}}
\]

Then, using \( V_b = 6 \text{ eV} \), \( L = 700 \text{ pm} \), \( mc^2 = 511,000 \text{ eV} \), and \( \ln 10^{-3} = -6.907 \), we have

\[
E = V_b - \frac{(197,500 \text{ eV} \cdot \text{pm} \cdot 6.907)^2}{2 \times 511,000 \text{ eV}} = 5.07 \text{ eV}
\]

6. Angular Momentum
If an electron in a hydrogen atom is in a state with \( l = 5 \), what is the smallest possible angle between \( \vec{L} \) and the \( z \)-axis?

**SOLUTION:**
The largest possible value of \( L_z \) for \( l = 5 \) is \( L_z = m \hbar = 5 \hbar \). The magnitude of \( \vec{L} \) is \( L = \sqrt{l(l+1)} \hbar = \sqrt{30} \hbar \). The angle between \( \vec{L} \) and its projection on the \( z \)-axis, \( L_z \), is given by

\[
\cos \theta = \frac{L_z}{L} = \frac{L_z}{\sqrt{30} \hbar}
\]

This angle is the smallest when \( L_z \) takes on its largest value, \( 5 \hbar \). Then

\[
\cos \theta = \frac{5 \hbar}{\sqrt{30} \hbar} = \frac{5}{\sqrt{30}} = 0.913
\]

which gives \( \theta = 24.1^\circ \).
7. Quantum States of the Hydrogen Atom

A hydrogen atom state has a maximum $m_l$ value of 4. What can you say about the rest of its quantum numbers?

**SOLUTION:**

The maximum value of $m_l$ equals the value of $l$ for the state, so $l = 4$. If $l = 4$, then since $l$ ranges up to $n-1$, we must have $n \geq 5$. The quantum number $m_s = \pm \frac{1}{2}$ always.

8. FOR EXTRA CREDIT: Heisenberg Uncertainty Principle

a) You are dropping marbles, of mass 30 g, from the roof of a building, onto a small target 20 m below. You are trying to be as accurate as possible in hitting the target, and are concerned that the Heisenberg Uncertainty Principle may limit your accuracy. Using the uncertainty relation in the form $\Delta x \Delta p_x \geq \frac{\hbar}{2}$, estimate the distance by which you will miss the target due to this effect.

**SOLUTION:**

The figure below shows the arrangement.

At the roof, you do the best job you can to line up the marble with the target, achieving a localization of the marble equal to $\Delta x_i$. Because of the uncertainty principle, this localization causes an uncertainty in the $x$-component of the momentum given by

$$\Delta p_x \geq \frac{\hbar}{2 \Delta x_i}.$$ 

This, in turn, means that the $x$-velocity of the marble has an uncertainty

$$\Delta v_x = \frac{\Delta p_x}{m} \geq \frac{\hbar}{2m \Delta x_i},$$

in which $m$ is the mass of the marble. The time which it takes the marble to fall from the roof to the target is given by

$$t = \sqrt{\frac{2H}{g}}.$$
where \( H \) is the height of the roof and \( g \) is the acceleration of gravity. Because of the \( x \)-velocity variation \( \Delta v_x \) caused by the uncertainty principle, there will be a variation of where the marble hits the ground of

\[
\Delta x_f = \Delta v_x l = \Delta v_x \sqrt{\frac{2H}{g}} \geq \frac{\hbar}{2m\Delta x_i} \sqrt{\frac{2H}{g}}.
\]

The total error in where the marble hits the ground is then given by

\[
\Delta x_{total} = \Delta x_i + \Delta x_f \geq \Delta x_i + \frac{\hbar}{2m\Delta x_i} \sqrt{\frac{2H}{g}}.
\]

This says that as we decrease \( \Delta x_i \), the second term, due to \( \Delta p_x \), gets larger. There is some optimum value of \( \Delta x_i \) which corresponds to a minimum value of \( \Delta x_{total} \). We find this by minimizing \( \Delta x_{total} \) as a function of \( \Delta x_i \). The value of \( \Delta x_i \) that minimizes \( \Delta x_{total} \) is the solution to the equation

\[
d\Delta x_{total} \over d\Delta x_i = 0.
\]

Carrying out the indicated differentiation, we get

\[
\Delta x_{i, minimum} \geq \sqrt{\frac{\hbar}{2m g}} \left( \frac{2H}{g} \right)^{1/4}.
\]

Then, plugging this value in to the equation for \( \Delta x_{total} \) gives

\[
\Delta x_{total, minimum} \geq 2 \sqrt{\frac{\hbar}{2m g}} \left( \frac{2H}{g} \right)^{1/4}.
\]

Evaluating this with the given numbers yields

\[
\Delta x_{total, minimum} \geq 2 \sqrt{\frac{6.63 \times 10^{-34} \text{ J s}}{4\pi \times 0.03 \text{ kg}}} \left( \frac{2 \times 20 \text{ m}}{9.8 \text{ m/s}^2} \right)^{1/4} = 1.2 \times 10^{-16} \text{ m}.
\]

This distance is a few percent of the diameter of the nucleus of a uranium atom. So there is no cause for concern about errors due to the uncertainty principle in this case.

b) You decide to change from dropping marbles to dropping helium atoms (mass=\(6.64 \times 10^{-27} \text{ kg}\)). What is your estimated miss distance in this case?