Physics 214 Spring 99—Problem Set 12—Optional Problems

Handout April 20, 1999

1. Time-dependent Schrödinger Equation

a) Show that \( \Psi(x,t) = Ae^{ikx-\omega t} \) satisfies the wave equation for a string:

\[
\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi(x,t)}{\partial t^2}.
\]

b) Show that the same wave function \( \Psi(x,t) = Ae^{ikx-\omega t} \) satisfies the time-dependent Schrödinger equation with \( V(x,t) = 0 \):

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} = i\hbar \frac{\partial \Psi(x,t)}{\partial t}.
\]

SOLUTION: In each case, we take the derivatives of \( \Psi(x,t) \) and see if the left hand side equals the right hand side of the equation for some choice of constants.

a) The space derivative is

\[
\frac{\partial^2 \Psi(x,t)}{\partial x^2} = -k^2 \Psi(x,t)
\]

The time derivative is

\[
\frac{\partial^2 \Psi(x,t)}{\partial t^2} = -\omega^2 \Psi(x,t)
\]

For \( \omega/v = k \), the left and right sides of the equation are equal, and so \( \Psi(x,t) \) is a solution.

b) The space derivative is

\[
\frac{\partial^2 \Psi(x,t)}{\partial x^2} = -k^2 \Psi(x,t)
\]

The time derivative is

\[
\frac{\partial \Psi(x,t)}{\partial t} = -i\hbar \Psi(x,t)
\]

For \( \hbar^2/2m = \hbar\omega \), the left and right sides of the equation are equal, and so \( \Psi(x,t) \) is a solution.

2. Probability Density

At time \( t = 0 \), the wave function of an electron is \( \Psi(x,0) = Ae^{-x^2/2b^2} \).

a) What is the probability density \( P(x) \) for this electron, in terms of \( A \) and \( b \)?

b) What must \( A \) be for this wave function to satisfy the normalization condition? (Note: you'll need to refer to a table of definite integrals to solve this problem).

c) What is the probability of finding the electron to the right of the origin (i.e., at some position \( x > 0 \))? Use the normalization condition to eliminate \( A \). (Hint: Either sketching \( P(x) \) or using Taylor's theorem may help with this one).

d) What is the probability of finding the electron somewhere in the interval \( 0 < x < \delta x \) where \( \delta x << b \)? Use the normalization condition to eliminate \( A \). (Hint: Either sketching \( P(x) \) or using Taylor's theorem may help with this one).

SOLUTION:

a) \( P(x) = |\Psi(x)|^2 = A^2e^{-x^2/2b^2} \) (this second equality is true because \( \Psi \) is real). Squaring the wave function gives \( P(x) = A^2e^{-x^2/2b^2} \).

b) The normalization condition, which says that the probability of finding the electron somewhere is 1, is

\[
\int_{-\infty}^{+\infty} P(x)dx = 1
\]

Putting in the \( P(x) \) from the previous part,

\[
A^2 \int_{-\infty}^{+\infty} e^{-x^2/2b^2} dx = A^2b\sqrt{\pi} = 1,
\]

so \( A = \frac{1}{b\sqrt{\pi}} \).

c) The probability density is an even function of \( x \) (i.e., \( P(x) = P(-x) \)), so the electron is equally likely to be to the left or the right of the origin. The probability of finding the electron somewhere is 1, so the probability of finding it on the right side is 1/2.
3. Particle in a Box

An electron is in a 1-dimensional well with infinitely high sides and width $L$. Which of the following statements must be true for a standing wave solution?

(A) If $L$ decreases a factor of 10, the ground state energy will increase by 100.

(B) If $L$ increases a factor of 10, the ground state energy will increase by 100.

(C) The probability of finding the electron in the box has a maximum at the center of the box.

(D) The probability of finding the electron outside the box is not identically zero.

(E) Two of the above answers are correct.

(F) None of the above answers are correct.

SOLUTION: (D) is not true because the box has infinitely high walls, (C) is not true because the electron wave function (if it is not in the ground state but in the first excited state for example) can have a node in the center of the box, (B) is not true since as the box width increases the ground state energy will decrease, and (A) is the correct answer because in fact, the ground state energy is proportional to $1/L^2$.

d) $P(0) = A^2 = \frac{1}{\sqrt{2}}$, and $P(x)$ is very nearly constant for $0 < x < \delta x$ where $\delta x << b$. A way of seeing this is to use Taylor's theorem:

$$P(\delta x) \approx P(0) + \delta x \frac{dP}{dx}(0) + \frac{\delta x^2}{2} \frac{d^2 P}{dx^2}(0) = P(0) - \frac{\delta x^2}{b^2} \approx P(0).$$

Then

$$\text{Probability} = \int_0^{\delta x} P(x) dx \approx \frac{\delta x}{b\sqrt{\pi}}.$$ 

4. Electron in a Box

An electron is confined in a 1-dimensional potential box by a potential $V(x)$, where $V(x) = 0, -a < x < a$ and $V(x) \to \text{infinity}, x > a$ and $x < -a$.

a) Inside the box in the ground state, the wavefunction is given by $R_1(x) = A_1 \cos(k_1 x)$ where $A_1, k_1$ are constants. Using the boundary conditions that the wavefunction is zero at $x = \pm a$ find an expression for $k_1$.

SOLUTION: Setting $R_1(\pm a) = 0$ at $x = \pm a$ gives the relation $k_1 a = \pi/2$ or $k_1 = \pi/(2a)$.

b) The wavefunction for the nth energy level can be written as $R_n(x) = A_n \sin(k_n x) + B_n \cos(k_n x)$. What are $B_2$ and $k_2$ for the first excited state $n = 2$?

SOLUTION: The first excited state has a single interior node at $x = 0$ which gives $B_2 = 0$. To satisfy the boundary conditions $R_2(\pm a) = 0$ at $x = \pm a$, we then have $A_2 \sin(k_2 a) = 0$ which implies $k_2 a = \pi$ or $k_2 = \pi/a$.

c) The electron is initially in its ground state. Then white light consisting of a wide range of frequencies illuminates the electron which can absorb photons and jump to excited energy states. What are the two lowest energies of photons that can be absorbed by the electron in transitions from the ground state? Express your answers in terms of the ground state energy $E_0$.

SOLUTION: We know for the square well potential, $E_n = \hbar^2 n^2 / (8m a^2)$. The lowest energy photon that can be absorbed in the ground state is the one that induces the transition for the electron from $E_1$ to $E_2$. $\Delta_{1\to2} = E_2 - E_1 = 4E_0 - E_1 = 3E_0$ where $E_0 = \hbar^2 / (32ma^2)$. The next lowest energy photon the electron in the ground state can absorb is $\Delta_{1\to3} = E_3 - E_1 = 9E_0 - E_1 = 8E_0$ where $E_1 = \hbar^2 / (32ma^2)$.

d) In real life the potential can’t be infinite outside the box. Suppose that $V(x) = V_0$ for $x > a$ and $x < -a$ where $V_0$ is large but finite. Will the energy of the first excited state for the finite box be less than, greater than, or equal to the value calculated for the potential well with infinite walls?

SOLUTION: The first excited state energy for the finite box will be less. The reason is that the wavefunction no longer has to go to zero at $x = \pm a$.
and can leak out of the box. The electron is effectively confined to a larger box and so its momentum spread and its energy will be less. Another way to think about it is that the wavelength of the wave function is larger since the particle can leak into previously forbidden regions; since the energy is inversely proportional to the square of the wavelength, the energy will be less.

5. Particle in a Box: Probabilities

For a particle in a 1-dimensional box of width $L$ and infinitely high sides,

a) estimate the probability that in the lowest energy state, the particle is in the interval $L/4$ to $3L/4$.

b) estimate the probability that the $n=3$ excited state and the $n=19$ excited state are in the interval $L/4$ to $3L/4$.

c) Justify the answers to parts (a) and (b) (hint: draw a picture and explain qualitatively why your answers came out the way that they did).

**SOLUTION:** For a particle in a box,

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right).$$

In general, the probability to find the particle in the interval $a$ to $b$ is

$$\int_a^b \psi_n^2(x) dx = \frac{2}{L} \int_a^b \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \left( \frac{1}{2} - \frac{L}{4n\pi} \sin\left(\frac{2n\pi x}{L}\right) \right)_a^b.$$

a) For $n = 1$, the above formula becomes:

$$\int_{L/4}^{3L/4} \psi_1^2(x) dx = \frac{2}{L} \left( \frac{3L}{8} - \frac{L}{8} - \frac{L}{4\pi} \sin\left(\frac{3\pi}{2}\right) + \frac{L}{4\pi} \sin\left(\frac{\pi}{2}\right) \right) = \frac{1}{2} + \frac{1}{\pi}.$$

b) For $n = 3$:

$$\int_{L/4}^{3L/4} \psi_3^2(x) dx = \frac{2}{L} \left( \frac{3L}{8} - \frac{L}{8} - \frac{L}{12\pi} \sin\left(\frac{18\pi}{4}\right) + \frac{L}{12\pi} \sin\left(\frac{6\pi}{4}\right) \right) = \frac{1}{2} - \frac{1}{3\pi}.$$

And for $n = 19$:

$$\int_{L/4}^{3L/4} \psi_{19}^2(x) dx = \frac{2}{L} \left( \frac{3L}{8} - \frac{L}{8} - \frac{L}{(4\pi)(19)} \sin\left(\frac{6(19)\pi}{4}\right) + \frac{L}{(4\pi)(19)} \sin\left(\frac{2(19)\pi}{4}\right) \right) = \frac{1}{2} - \frac{1}{19\pi}.$$

c) For the lowest energy state the wavefunction squared is shown below and clearly there is a higher probability (area under the $P(x)$ curve) to find the particle in the center of the box since $\psi(x)$ is largest there.
SOLUTION:

The time independent Schrödinger equation in 1 dimension is given by

\[
\frac{d^2 \psi}{dx^2} = -\frac{2m}{\hbar^2} (E - U) \psi
\]

In the region of interest in our problem \( E = 0 \) and hence rearranging yields

\[
U = \frac{\hbar^2}{2m} \frac{1}{\psi} \frac{d^2 \psi}{dx^2}
\]

Hence

\[
\frac{d\psi}{dx} = Ae^{-\frac{x^2}{2L^2}} - 2A \frac{x^2}{L^2} e^{-\frac{x^2}{2L^2}}
\]

\[
\frac{d^2 \psi}{dx^2} = -2A \frac{x}{L^2} e^{-\frac{x^2}{2L^2}} - 4A \frac{x^2}{L^4} e^{-\frac{x^2}{2L^2}} + 4A \frac{x^3}{L^4} - \frac{x^2}{L^2}
\]

\[
= A \frac{x^2}{L^2} e^{-\frac{x^2}{2L^2}} \left( \frac{4x^2}{L^2} - 6 \right) \psi(x)
\]

Thus

\[
U(x) = \frac{\hbar^2}{2mL^2} \left( \frac{4x^2}{L^2} - 6 \right)
\]

(b) A parabola with \( U(0) = -\frac{3\hbar^2}{mL^2} \) and zeros located at \( x = \pm \sqrt{\frac{3}{2}} L \).

7. Uncertainty Principle in Waves

a) Waves on a String
Suppose we want to create a localized wave on a string, as a superposition of standing waves. Let’s try to make all the standing waves add up to a sharp peak in the middle of the string. Consider a string that stretches from \( x = -L/2 \) to \( L/2 \). (That is, \( x = 0 \) is at the center of the string where we want the peak, and not at the edge.) The standing waves corresponding to all odd harmonics (the fundamental, the third harmonic, the fifth harmonic) have peaks at \( x = 0 \): these have the form (in our variables)

\[
y_{0m-1}(x) = \cos(\pi(2m-1)x/L) = \cos(k_{2m-1}x)
\]

where \( m = 1, 2, 3, \ldots \) and \( k_n = \pi n / L \) is called the wave number of the standing wave. So, to get a big peak in the middle, we want to add up \( M \) waves

\[
y(x) = \sum_{m=m_0}^{M+m_0-1} \cos(\pi(2m-1)x/L) = \sum_{m=m_0}^{M+m_0-1} \cos(k_{2m-1}x)
\]

Make a plot, with \( L = 6 \), using either a spreadsheet program or MatLab, from \( x = -L/2 \) to \( x = +L/2 \) in intervals of 0.1, of \( y(x) \), for \( M = 2, 5, 10 \) and 20. To see the wave envelope clearly, start the sum at \( m_0 = 40 \).

**SOLUTION:**
Find the width of the peaks $\Delta x$, defined as the distance from $x = 0$ to the first zero in the envelope of the transverse displacement $y$, and fill in the following table:

<table>
<thead>
<tr>
<th>$M$</th>
<th>$\Delta x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>??</td>
</tr>
<tr>
<td>5</td>
<td>??</td>
</tr>
<tr>
<td>10</td>
<td>??</td>
</tr>
<tr>
<td>20</td>
<td>??</td>
</tr>
</tbody>
</table>

**SOLUTION:**

<table>
<thead>
<tr>
<th>$M$</th>
<th>$\Delta x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>1.2</td>
</tr>
<tr>
<td>10</td>
<td>0.6</td>
</tr>
<tr>
<td>20</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Notice that as we use larger and larger ranges of wave numbers $k_n$ we get a narrower and narrower peak! The wave numbers range from $k \approx 2\pi m_0/L$ to $k \approx 2\pi (M + m_0)/L$, so the spread $\Delta k \approx 2\pi M/L$. Show that we have a position-wave number uncertainty relation

$$\Delta x \Delta k \gtrsim C.$$ 

What is the constant $C$? The constant $C$ depends on our definition of the widths $\Delta x$ and $\Delta x$. (The uncertainty relations discussed in Serway are for the root-mean-square widths. We use full widths in this problem.)

**SOLUTION:**

We notice from our table above that $M \Delta x \sim L$. Hence

$$\Delta x \Delta k \sim \frac{L}{M} \times \pi \frac{2M}{L} = 2\pi$$

Suppose, instead of waves on a string, these waves were quantum mechanical probability waves associated with a particle of momentum $p = \frac{h}{\lambda} = h k$. By using the de Broglie formula to get the range of momenta from the range in wave numbers, show that it satisfies the position-momentum uncertainty relation

$$\Delta x \Delta p \gtrsim h.$$ 

**SOLUTION:**

The de Broglie formula $p = \hbar k$ means that the spread in momentum is $\Delta p = h \Delta k$. From the previous section

$$\Delta x \Delta k \gtrsim 2\pi \quad \Rightarrow \quad \Delta x \frac{\Delta p}{\hbar} \gtrsim 2\pi \quad \Rightarrow \quad \Delta x \Delta p \gtrsim 2\pi \hbar = h$$

**b) Making a laser pulse**

Let’s think about a laser beam: nearly monochromatic light. Lasers are not quite monochromatic... Let’s model a laser in a way nearly the
same as our calculation above: numerically add up some cosine waves spread in frequency between \( f_0 - \Delta f/2 \) and \( f_0 + \Delta f/2 \):

\[
E(t) = \sum_{m=-M/2}^{M/2} \cos \left( 2\pi \left( f_0 + \frac{m}{M+1} \Delta f \right) t \right)
\]

Note that this is just the generalization of beats, with \( M \) waves of different frequencies, rather than just two. Make a plot, with \( f_0 = 5 \) and \( \Delta f = 0.5 \), using either a spreadsheet program or MatLab, from \( t = -4 \) to \( t = 4 \) in intervals of 0.04, of \( E(t) \), with \( M = 10 \). How wide is the pulse in time \( \Delta t \), i.e. from \( t = 0 \) to the first zero of the envelope? (Note \( \Delta t \) should be the same if measured from that first zero to the second zero in the envelope).

Show that it satisfies the frequency–time uncertainty relation

\[ \Delta f \Delta t \gtrsim 1. \]

**SOLUTION:**

We note from the figure \( \Delta t = 2 \). Hence

\[ \Delta f \Delta t = 0.5 \times 2 \gtrsim 1. \]

Using the energy formula for a photon, show that the energy spread in this pulse \( \Delta E \) satisfies the energy–time uncertainty relation

\[ \Delta E \Delta t \gtrsim h. \]

**SOLUTION:**

Since \( \Delta f = \Delta E/h \) and hence

\[ \Delta f \Delta t = \frac{\Delta E}{h} \Delta t \gtrsim 1 \quad \Rightarrow \quad \Delta E \Delta t \gtrsim h \]