Physics 214 Spring 99—Problem Set 12—Solutions
Handout April 29, 1999

1. Reading Assignment

Reading Assignment:
Week Beginning April 19: Serway 41.4, 41.6, 41.7, 41.8, 41.11
Week Beginning April 26: Serway 41.9, 41.10, 42.1, 42.2, 42.3, 42.4, 42.5

2. Particle in a Box and the Uncertainty Principle

A particle of mass $m$ is confined to a region of the $x$-axis extending from $x = -L$ to $x = L$. The potential $V = 0$ in this region. A properly normalized solution to the time-independent Schrödinger equation for the particle has the form

$$\psi_n(x) = \sqrt{\frac{1}{L}} \cos k_n x,$$

in which $k_n = \pm \frac{(2n-1)\pi}{2L}$, and $n = 1, 2, 3, \ldots$

a) We consider the system at $t = 0$, so that we can ignore the time-dependent part of the wavefunction $e^{-i\frac{E}{\hbar}t}$. Since the wavefunctions obey the principle of superposition, any linear combination of wavefunctions is also a solution to the Schrödinger equation. Consider quantum states of very large $n$, for which the change in $n$ from one state to the next is much smaller than $n$, so that $n$ (and $k_n = \pm \frac{(2n-1)\pi}{2L}$) is essentially a continuous variable. Construct the superposition of all wavefunctions, with $k$ running from $k = -\Delta k/2$ to $k = \Delta k/2$, by doing the integral:

$$\psi_{\text{total}}(x) = \frac{1}{\Delta k} \int dk \psi_n(x) = \frac{1}{\Delta k \sqrt{L}} \int \frac{dk}{\Delta k} \cos (kx)$$

b) Square the resulting wavefunction to get the probability density, and compare with the equation for the intensity in a single slit diffraction pattern,

$$I(\theta) = I_{\text{max}} \frac{\sin^2 \alpha}{\alpha^2}, \quad \alpha = \frac{\pi a}{\lambda} \sin \theta$$

What is the equivalent of $\alpha$, in terms of $x$? You know that the first diffraction minimum occurs at $\alpha = \pi$. Find $x_{\text{min}}$, the value of $x$ this corresponds to.

For the specific case of $L = 10$ nm, and $\Delta k = 8\pi / L$, plot the probability density as a function of $x$. What is the numerical value of $x_{\text{min}}$ for this case?

SOLUTION:

The probability density is

$$P_{\text{total}}(x) = |\psi_{\text{total}}(x)|^2 = \frac{1}{L} \sin^2 \left( \frac{\Delta k x}{2} \right) \left( \frac{\Delta k}{2} \right)^2$$

The equivalent of $\alpha$ is $(\Delta k x)/2$. Setting this equal to $\pi$ gives $x_{\text{min}} = \frac{2\pi}{\Delta k}$.

The probability density is plotted below:
c) From the plot in part (b), and from the similarity in the form of the equation, you can see that the probability density \( P_{total} \) has the same form as a diffraction pattern, with the first minimum at \( x_{min} \). Thus, the particle is localized to within \( \Delta x = 2x_{min} \). Show that this is consistent with the Heisenberg Uncertainty Principle.

**SOLUTION:**

Using the result from part (b), we have

\[
\Delta x = 2 \frac{2\pi}{\Delta k}
\]

But \( k = \frac{2\pi}{\lambda} = \frac{2\pi p}{\hbar} \), so

\[
\Delta k = \frac{2\pi \Delta p}{\hbar}
\]

Then

\[
\Delta x = 2 \frac{\hbar}{\Delta p}
\]

which is consistent with the uncertainty relation

\[\Delta x \Delta p \geq \hbar.\]
b) For the first excited state, of energy $E_2$, where is it most probable to find the electron?

**SOLUTION:** It is most probable to find the electron at $x = \pm L/2$.

c) How will the energy of the lowest energy state change if the width of the box is reduced by a factor of 2?

**SOLUTION:** The energy varies like $1/L^2$, so it will increase a factor of 4.

d) What is the probability that you will find the electron between $-L/2$ and $+L/2$ when the electron is in energy state $E_n$ and $n$ is even?

**SOLUTION:** An integral number of half-wavelengths must fit in the box in order to satisfy the boundary conditions. Thus $n\lambda_n/2 = 2L$, so $\lambda_n = 4L/n$ and $k_n = 2\pi/\lambda_n = n\pi/(2L)$. In addition, the wavefunction must go to zero at $x = \pm L$. The solutions are then $\psi_n(x) = B\cos\left(\frac{n\pi x}{2L}\right)$ for $n$ odd, and $\psi_n(x) = A\sin\left(\frac{n\pi x}{2L}\right)$ for $n$ even. We are concerned in this section with the solutions for $n$ even. What is $A$? It comes from the normalization condition:

$$\int_{-\infty}^{\infty} |\psi_n(x)|^2 dx = 1,$$

so

$$\int_{-L}^{L} |\psi_n(x)|^2 dx = A^2\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{2L}\right) dx = A^2L = 1,$$

so $A = 1/\sqrt{L}$. The probability of finding the particle between $-L/2$ and $L/2$ is the integral of the probability density over that range:

$$\int_{-L/2}^{L/2} |\psi_n(x)|^2 dx = A^2\int_{-L/2}^{L/2} \sin^2\left(\frac{n\pi x}{2L}\right) dx$$

$$= \frac{1}{L} \left[ \frac{L}{2} - \frac{L}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right] = \frac{1}{2}$$

for $n$ even.

e) If the electron is in its ground state, (with energy $E_1$), what is the probability that it will be found in the interval between $x = 0$ and $x = \Delta x$, where $\Delta x << L$?

**SOLUTION:** The form of the wavefunction for $n=1$ is $\psi_1(x) = B\cos\left(\frac{\pi x}{2L}\right)$, where $B$ is given from the normalization condition as

$$\int_{-L}^{L} |\psi_1(x)|^2 dx = B^2\int_{-L}^{L} \cos^2\left(\frac{\pi x}{2L}\right) dx = B^2L = 1,$$

so $B = 1/\sqrt{L}$. The probability that the electron will be found in the interval between $x = 0$ and $x = \Delta x$, is

$$\int_{0}^{\Delta x} |\psi_1(x)|^2 dx = \frac{1}{L} \int_{0}^{\Delta x} \cos^2\left(\frac{\pi x}{2L}\right) dx \approx \frac{\Delta x}{L}$$

for $\Delta x << L$.

4. Serway, Chapter 41, pg 1247, Problem 28
SOLUTION: The wave function is given by

\[ \psi = \sqrt{\frac{2}{L}} \sin \left( \frac{\pi x}{L} \right) \]

(a) The probability is determined from the integral

\[
P(0 \leq x \leq \frac{L}{3}) = \int_{0}^{\frac{L}{3}} dx \mid \psi \mid^2 = \frac{2}{L} \int_{0}^{\frac{L}{3}} dx \sin^2 \left( \frac{\pi x}{L} \right)
\]

\[
= \frac{2}{L} \int_{0}^{\frac{L}{3}} dx \left( \frac{1}{2} - \frac{1}{2} \cos \left( \frac{2\pi x}{L} \right) \right)
\]

\[
= \frac{1}{L} \left[ \frac{x}{2} - \frac{1}{2\pi} \sin \left( \frac{2\pi x}{L} \right) \right]_0^{\frac{L}{3}}
\]

\[
= \frac{1}{3} - \frac{1}{2\pi} \sin \left( \frac{2\pi}{3} \right)
\]

\[
= \frac{1}{3} - \frac{\sqrt{3}}{4\pi} = 0.196
\]

(b) Note \( \mid \psi(x) \mid^2 \) is symmetric about \( x = \frac{L}{2} \) and by definition \( P(0 \leq x \leq L) = 1 \). Hence

\[
P(0 \leq x \leq \frac{L}{3}) = P(\frac{2L}{3} \leq x \leq L)
\]

and therefore

\[
P(\frac{L}{3} \leq x \leq \frac{2L}{3}) = 1 - 2 \times 0.196 = 0.608
\]

5. Infinite Well with rounded edges

A particle of mass \( m \) moves in a potential well of width \( 2L \) (from \( x = -L \) to \( x = L \)), and in this well the potential is given by

\[ U(x) = \begin{cases} 
\infty & x < -L \\
\frac{h^2}{mL^2} \left( \frac{x^2}{L^2} - x^2 \right) & -L \leq x \leq L \\
\infty & x > L
\end{cases} \]

In addition, the particle is in a stationary state described by the wave function

\[
\psi(x) = \begin{cases} 
0 & x \leq -L \\
A \left(1 - \frac{x^2}{L^2}\right) & -L < x < L \\
0 & x \geq L
\end{cases}
\]

(a) Determine the energy of the particle in terms of \( h \) and \( m \).

SOLUTION: The Time-independent Schrödinger equation is given by

\[
\frac{d^2\psi}{dx^2} = -\frac{2m}{h^2} (E - U) \psi
\]

Hence for the given \( \psi(x) \) we have

\[
\frac{d^2\psi}{dx^2} = -\frac{2m}{h^2} (E - U) \psi
\]

\[
-2 \frac{A}{L^2} = -\frac{2m}{h^2} \left( E + \frac{h^2}{mL^2} \left( \frac{x^2}{L^2} \right) \right) A \left(1 - \frac{x^2}{L^2}\right)
\]

\[
-2 \frac{A}{L^2} = \frac{2m}{h^2} \left( E + \frac{h^2}{mL^2} \left( \frac{x^2}{L^2} \right) \right) A \left(1 - \frac{x^2}{L^2}\right)
\]

Solving for \( E \) yields \( E = \frac{h^2}{mL^2} \)

(b) Determine the numerical value of \( A \).

SOLUTION: The normalization condition is given by

\[
\int_{-\infty}^{\infty} \mid \psi(x) \mid^2 dx = 1
\]

Hence

\[
1 = A^2 \int_{-\infty}^{\infty} (1 - \frac{x^2}{L^2})^2 dx = 2A^2 \int_{0}^{L} (1 - \frac{x^2}{L^2} + \frac{x^4}{L^4}) dx
\]

\[
= 2A^2 \left[ x - \frac{x^3}{3L^2} + \frac{x^5}{5L^4} \right]_0^L
\]

\[
= 2A^2 \left( L - \frac{2L^3}{3L^2} + \frac{L^5}{5L^4} \right)
\]

\[
= \frac{168}{15} A^2 L
\]
Thus \[ A = \sqrt{\frac{15}{16L}}. \]

c) Determine the most probable location(s) of the particle.

**SOLUTION:** The probability density \( P(x) = |\psi(x)|^2 = A^2(1 - \frac{x^2}{L^2})^2. \)
Maxima and minima occur when 
\[
\frac{dP(x)}{dx} = -4x \left( 1 - \frac{x^2}{L^2} \right) = 0
\]
This happens at \( x = \pm L \) and \( x = 0 \). But \( P(\pm L) = 0 \), so the most probable location for the particle is \( x = 0 \).

6. Time-dependent Schrödinger Equation

a) By substitution, show that, for any non-zero values of \( \omega, k, \) and \( A \), the wave function \( \Psi(x, t) = A\sin(kx - \omega t) \) does not satisfy the time-dependent Schrödinger equation, with \( V(x, t) = 0 \).

**SOLUTION:**

The time-dependent Schrödinger equation, with \( V(x, t) = 0 \), is
\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} = i\hbar \frac{\partial \Psi(x, t)}{\partial t}
\]
The space derivative is
\[
\frac{\partial^2 \Psi(x, t)}{\partial x^2} = -k^2 A\sin(kx - \omega t)
\]
The time derivative is
\[
\frac{\partial \Psi(x, t)}{\partial t} = \omega A\cos(kx - \omega t)
\]
Substituting, we have
\[
k^2 \frac{\hbar^2}{2m} A\sin(kx - \omega t) \neq i\hbar \omega A\cos(kx - \omega t)
\]
Since sines and cosines are linearly independent functions, and since purely real numbers cannot equal purely imaginary numbers, the Schrödinger equation is not satisfied, for arbitrary \( x \) and \( t \).

b) The wave function \( \Psi(x, t) = Ae^{i(kx-\omega t)} \) satisfies the time-dependent Schrödinger equation. Explain why it does not describe a particle in a bound (standing wave) state of an infinite potential well. What wave function must be added to the above to obtain a superposition which does describe a bound state?

**SOLUTION:**

The wave function \( Ae^{i(kx-\omega t)} \) corresponds to a traveling wave, moving in the \(+x\) direction. This is not a standing wave solution. To obtain a standing wave, we must add a traveling wave moving in the \(-x\) direction, that is, \( Ae^{i(-kx-\omega t)} \). The resulting superposition
\[
\Psi_{\text{total}}(x, t) = Ae^{i(kx-\omega t)} + Ae^{i(-kx-\omega t)} = 2A \cos kxe^{-\omega t}
\]
is a standing wave and describes a bound state.

7. The Wavefunction of an Electron

A sinusoidal one-dimensional traveling wavefunction for an electron is given by
\[
\Psi(x, t) = Ae^{i(1.5 \times 10^{16} \text{ m}^{-1}x - 3 \times 10^{16} \text{ s}^{-1}t)}.
\]
a) What is the momentum of the electron?

**SOLUTION:** We know that the general form of a sinusoidal one-dimensional traveling wavefunction is
\[
\Psi(x, t) = Ae^{ipx-\beta t}.
\]
So, by comparison with the above relation, we have
\[
P = \frac{p}{\hbar} = 1.5 \times 10^{10} \text{ kg m/s}
\]
which gives \( p = 1.58 \times 10^{-24} \text{ kg m/s} \).

b) What is the total energy of the electron (in eV)?

SOLUTION:

Again, by comparison with the relation in part (a), we have

\[ \frac{E}{\hbar} = 3 \times 10^{16} \text{ s}^{-1} \]

which gives \( E = 3.16 \times 10^{-18} \text{ J} = 19.8 \text{ eV} \).

c) What is the potential energy of the electron (in eV)?

SOLUTION:

From part (a), we know the momentum. The kinetic energy is given by

\[ K = \frac{p^2}{2m} = 1.37 \times 10^{-18} \text{ J} = 8.6 \text{ eV} \]

using \( m = 9.11 \times 10^{-31} \text{ kg} \). Using \( E \) from part (b), we have \( V = E - K = 11.2 \text{ eV} \).