Physics 214 Fall 98—Problem Set 5—[Total Points: 12]

Due Before 9:00 am October 1 1998

1. Reading Assignments from Serway
   Week Beginning September 22: 17.0 - 17.3, 17.5
   Week Beginning September 29: 34.0 - 34.4, 34.7; 35.0 - 35.2; 37.5 - 37.7

2. Serway, Chapter 18, pg 523, Problem 42

3. Serway, Chapter 18, pg 524, Problem 45

4. Serway, Chapter 17, pg 493, Problem 15 (modified)

   Answer the questions (a), (c) from the text **BUT** for the following
   sinusoidal sound wave
   \[ s(x, t) = (2.00 \, \mu m) \cos [(0.5 \, m^{-1}) x - (2500 \, s^{-1}) t] \]

5. Serway, Chapter 17, pg 495, Problem 34

**Computing** [9 Points]

1. Solving the Wave Equation Numerically, Part 1

   We want to analyse numerically the motion of a string of length \( L \) that is
   being shaken up and down at one end, and is free at the other. The string
   is initially straight and at rest, and its transverse motion is determined
   by the wave equation
   \[ \mu \frac{\partial^2 y(x, t)}{\partial t^2} = \tau \frac{\partial^2 y(x, t)}{\partial x^2} \]  (1.1)
   where

   \[ t = \text{time}, \]
   \[ x = \text{the distance measured along the string} \]
   \[ (\text{from } x = 0 \text{ to } x = L), \]
   \[ y(x, t) = \text{the string’s transverse displacement at} \]
   \[ \text{position } x \text{ and time } t, \]
   \[ \mu = \text{the mass density of the string}, \]
   \[ \tau = \text{the tension in the string}. \]

   Note that the separate values of the tension and mass density do not
   matter in this equation; only the ratio
   \[ v^2 \equiv \frac{\tau}{\mu} \]  (1.2)

   is important. The end of the string at \( x = L \) is free, while the motion of
   the other end is specified by some displacement function \( f(t) \):
   \[ y(0, t) = f(t) \quad \frac{\partial y(L, t)}{\partial x} = 0 \]  (1.3)

   for all time \( t \).

   To solve this problem, we must first understand exactly how the wave
   equation specifies the future motion of the string. Consider the left-hand
   side of the wave equation, Eq. (1.1). We can approximate the temporal
   second derivative by
   \[ \frac{\partial^2 y(x, t)}{\partial t^2} \approx \frac{y(x, t + \delta t) - 2y(x, t) + y(x, t - \delta t)}{\delta t^2} \]  (1.4)

   provided \( \delta t \) is not too large. Substituting this relation into the wave
   equation, we obtain an evolution equation equivalent to the wave equation:
   \[ y(x, t + \delta t) \approx 2y(x, t) - y(x, t - \delta t) + \delta^2 v^2 \frac{\partial^2 y(x, t)}{\partial x^2}. \]  (1.5)

   This equation tells us that we can predict the string displacement \( y \) at
   time \( t + \delta t \) from its values at the earlier times \( t \) and \( t - \delta t \), and from
   the spatial second derivative of the string—that is, \( \partial^2 y(x, t) / \partial x^2 \)—at time \( t \).

   Since the spatial second derivative can always be computed from \( y(x, t) \)
   (by differentiating), all we need to know to predict the future motion of
the string using the evolution equation is where the string is now \((y(x, t)\) for all \(x\)) and where it was in the immediate past \((y(x, t-\delta t)\) for all \(x\))—that is, all we need know are the initial conditions. Once \(y(x, t + \delta t)\) has been determined, the same procedure can be used again to compute \(y(x, t+2\delta t)\) but this time starting from \(y(x, t+\delta t)\) and \(y(x, t)\). By applying this procedure again and again, we can compute the position of the string at any later time.

A numerical study of this problem cannot possibly deal with all \(x\)'s; that would require infinite effort! Instead we will deal only with the displacements at \(N+1\) discrete points, \(x_i\), along the string:

\[
x_0 \equiv 0 \quad x_1 \equiv \delta x \quad x_2 \equiv 2\delta x \quad \ldots \quad x_N \equiv N\delta x = L.
\]

We can get a pretty good idea of what the whole string is doing from just the \(y(x_i, t)\)'s provided \(\delta x = L/N\) is small enough. What we need then is a modified evolution equation that involves only \(y\)'s at the points \(x_i\). We can determine the string's spatial second derivative at each point along the string \(x = x_i\) by using the analog of Eq. (1.4), but for space derivatives,

\[
\frac{\partial^2 y(x_i, t)}{\partial x^2} \approx \frac{y(x_{i+1}, t) - 2y(x_i, t) + y(x_{i-1}, t)}{\delta x^2}
\]

—which is valid again if \(\delta x\) is small enough.

You can then use the evolution equation above, Eq. (1.5), to find the next value of the displacement of each point on the string, using the spatial second derivative you just found. For this purpose we set \(x = x_i\) in Eq. (1.5) to obtain

\[
y(x_i, t + \delta t) \approx 2y(x_i, t) - y(x_i, t - \delta t) + (\nu\delta t)^2 \frac{\partial^2 y(x_i, t)}{\partial x^2},
\]

Thus you can compute \(y(x_i, t+\delta t)\) for all \(i = 1 \ldots N-1\) using the evolution equation and initial data for \(y(x_i, t)\) and \(y(x_i, t - \delta t)\) at all \(i\); and, once more, the procedure can be iterated to obtain the position of the string at any later time.

Note that the approximation for the spatial second derivative, Eq. (1.7), makes no sense when applied at the endpoints of the string. For example, to compute the spatial second derivative at \(x_0 = 0\) using this approximation requires \(y(x_{-1}, t)\)—the string displacement at \(x = -\delta x\), where there is no string. Consequently we are unable to use the wave equation to predict the motions of the ends of the string. Additional information is needed: the boundary conditions. In this problem we are allowing the end at \(x = L\) to be free, so \(\frac{\partial y(x, t)}{\partial x}\bigg|_{x=L} = 0\) for all \(t\); and we are shaking the end at \(x = 0\), so \(y(x_0, t) = f(t)\). The boundary condition at \(x = L\) needs further discussion. This boundary condition requires that the slope of the wave with position to be zero. The reflection of a wave with a free boundary can be viewed as the superposition of the actual wave with an imaginary same shape wave reflected in the boundary at \(x = L\) but moving in the opposite direction to the actual wave. So the value of the wave at \(L-\delta x \equiv x_{N-1}\) should be the same at \(L+\delta x \equiv x_{N+1}\) where \(x_{N+1}\) is an imaginary point but useful for the calculation, i.e.,

\[
y(x_{N-1}, t) = y(x_{N+1}, t) \quad \text{for all } t
\]

Thus the approximation for the spatial second derivative, Eq. (1.7), at \(x = L \equiv x_N\) can be written as

\[
\frac{\partial^2 y(x_N, t)}{\partial x^2} \approx \frac{y(x_{N+1}, t) - 2y(x_N, t) + y(x_{N-1}, t)}{\delta x^2}
\]

Hence using this for the spatial second derivative in the evolution equation, Eq. (1.8), we can calculate \(y(x_N, t+\delta t)\) and satisfy the boundary condition \(\frac{\partial y(x_N, t)}{\partial x}\bigg|_{x=L} = 0\) for all \(t\).

\section*{a) Prototyping a Numerical Solution:} Consider first a very crude analysis of string motion when one end of the string is suddenly jerked up and then back down again, making a short pulse in the string. Let the string have length \(L = 10\) m and wave speed \(v = 2\) m/s. Take \(\delta t = 1\) s and \(\delta x = 2\) m, and assume the initial conditions specified in the following table:
Fill in the column listing the spatial second derivative in this table. Use this data, together with the evolution equations, Eqs. (1.8–1.7), to compute all of the entries in such a table for \( t = \delta t \) and \( 2\delta t \). Assume that \( y \) vanishes at both ends of strings for all \( t \)'s later than \( t = 0 \). Is the pulse’s motion what you expected? Explain why or why not.

### 2. Solving the Wave Equation Numerically, Part 2

Write a MatLab program or use a spreadsheet program to compute string motion for a string of length \( L = 10 \) m, and wave speed \( v = 2 \) m/s. Try \( \delta x = 0.1 \) m and \( \delta t = 0.025 \) s, and set \( f(t) \), the displacement function for \( x = 0 \), equal to

\[
f(t) \equiv e^{-\frac{1}{2}(4-2t)^2} \quad \text{(cm)}.
\]

Use the evolution equations Eqs. (1.7–1.8) and the initial conditions:

\[
y(x_i,0) = y(x_i,-\delta t) = 0 \quad \text{for all } i = 1 \ldots N + 1.
\]

If you are running a spreadsheet program, this should be possible with a 103 column (displacement) \times 402 row (time) spreadsheet.

Hand in a listing of your MatLab program or a sample of your spreadsheet with your homework.

If you find your computer is very slow in solving this problem, or your spreadsheet is getting very large, then try using \( \delta x = 0.5 \) m and \( \delta t = 0.125 \) s to debug your programme and then go to the steps suggested above for your final runs.

Using your program, generate (and hand in) plots of string displacement \( y(x,t) \) versus position \( x \) for \( t = 2 \) s, \( 4 \) s, \( 6 \) s, \( 6.5 \) s, \( 7 \) s, \( 7.5 \) s, \( 8 \) s, and \( 10 \) s. Compute the speed of the pulse from your plots and compare your result with what is expected (from the **wave equation**). The pulse seems to be twice as high at \( t = 7 \) s. Explain why this happens.

Using your program, generate (and hand in) plots of the string transverse (chunk) velocity \( \frac{\partial y(x,t)}{\partial t} \) versus position \( x \) for \( t = 6 \) s and \( t = 8 \) s. Explain how your plots are consistent with the plots of displacement versus position: for example, show that the zero(s) of \( \frac{\partial y}{\partial t} \) is (are) correctly placed; and show that the magnitude and sign of the string’s transverse (chunk) velocity are consistent with the shape and speed of the pulse.