Physics 214 Fall 98—Problem Set 2—[Total Points: 9]  Due Before 9:00 am September 10 1998

Homework [3 Points]

1. Reading Assignments from Serway

   Week Beginning September 1: 16.1 - 16.4, 16.7, 16.9; 18.0 - 18.6
   Week Beginning September 8: 16.3 - 16.5, 16.8

2. Serway, Chapter 18, pg 526, Problem 68

3. Serway, Chapter 18, pg 524, Problem 46

4. Wave Equation

   Two waves given by
   
   \[ y_1(x,t) = A \cos(kx - \omega t) \]
   \[ y_2(x,t) = A \cos(kx + \omega t) \]

   interfere to produce a standing wave of the form
   
   \[ y(x,t) = y_1 + y_2 = 2A \cos kx \cos \omega t \]

   Hence verify that this standing wave is a solution to the wave equation given by
   
   \[ \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \].

5. Friction in the Wave Equation

   A stretched string is immersed in a viscous fluid. The string has mass per unit length \( \mu \) and is stretched to tension \( \tau \). We will derive a wave equation that takes account of the viscosity between the string and the fluid. Assume that the viscous force on a chunk of the string is proportional both to the length of the chunk and to its chunk velocity. (Think about why and when this might be a reasonable assumption.) Let \( \alpha \) be the proportionality constant.

   a) Draw a free body diagram (FBD) for a segment of string. It should look similar to the one used in class, with an extra force due to the viscosity.

   b) Hence derive the equation of motion for the string in a viscous fluid.

   c) If the viscous force is opposing the velocity, slowing it down, what is the sign of \( \alpha \) in your equation, positive or negative? What are the dimensions of \( \alpha \)?

Computing [6 Points]

1. Pendulum

   In this problem, you will solve the equation of motion for a pendulum of mass \( m \) and length \( l \) as derived in the text:

   \[ \frac{d^2 \theta(t)}{dt^2} = -\frac{g}{l} \sin \theta \quad (1.1) \]

   both analytically with the help of an approximation, and numerically using either a spreadsheet programme, e.g., Excel (hints by clicking on spreadsheet button at the PHYS 214 course Web Page, www.physics.cornell.edu/p214) or MatLab. In particular we wish to investigate the validity of the small angle approximation in regard to the true period of the motion.
a) The Analytic Solution for Small Angle Approximation

Show that for small displacements of the pendulum from its equilibrium position, the motion is approximately simple harmonic motion. **What is the frequency of the pendulum?** Write the general solution in this limit, and also the particular solution when $l = 10$ m, $m = 50.0$ gm, $\theta(0) = 0.3$ rad, and $\dot{\theta}(0) = 0$ rad/s.

b) Setting up the Numerical Solution

We now want to translate the exact equation of motion Eq. (1.1) for the pendulum into something that the computer can solve. In order to do this, we need to write the differentials in terms of discrete steps in time. Our starting point is the approximate expression for the derivative:

$$\dot{\theta}(t) \approx \frac{\theta(t + \Delta t/2) - \theta(t - \Delta t/2)}{\Delta t}$$

where a dot over a function indicates the derivative of that function with respect to time, two dots means the second derivative, and so on. Hence the second derivative is given by

$$\ddot{\theta}(t) \approx \frac{\dot{\theta}(t + \Delta t/2) - \dot{\theta}(t - \Delta t/2)}{\Delta t} = \frac{\dot{\theta}(t + \Delta t) - \dot{\theta}(t) - \dot{\theta}(t) + \dot{\theta}(t - \Delta t)}{\Delta t}$$

$$= \frac{\dot{\theta}(t + \Delta t) - 2\dot{\theta}(t) + \dot{\theta}(t - \Delta t)}{\Delta t^2}$$

Rearranging yields the expression

$$\theta(t + \Delta t) \approx \theta(t) + \Delta t \ddot{\theta}(t) + \frac{\Delta t^2}{2} \dot{\theta}(t) - \theta(t - \Delta t).$$

This says that if we know $\theta(t - \Delta t)$ and $\dot{\theta}(t)$, we can calculate $\theta(t + \Delta t)$ since we know $\dot{\theta}(t)$ in terms of $\theta(t)$ from the exact equation of motion Eq. (1.1).

There is one more hurdle that we need to overcome in order to use this to solve the differential equation. We see that Eq. (1.4) tells us that if we know the value of the function at time $t$ and at a previous time step $t - \Delta t$ then we can calculate the function at the next time step $t + \Delta t$. The problem is: How do we get started? The initial conditions for the pendulum problem specify $\theta(0)$ and $\dot{\theta}(0)$, not $\dot{\theta}(-\Delta t)$ which is what we need to plug into Eq. (1.4). We need to use a modified version of Eq. (1.2) to put the initial conditions in a useful form.

$$\theta(t - \Delta t) = \theta(t) - \dot{\theta}(t) \Delta t$$

(1.5)

When $t = 0$, this equation allows you to find $\dot{\theta}(-\Delta t)$.

For your pendulum, assume that $l = 10$ m and that initially, i.e. $t = 0$ s, that $\theta(0) = 0.3$ rad and $\dot{\theta}(0) = 0$ rad/s. To get a really concrete feel for how the differential equation works, we want you to step through a series of time steps constructing the solution with a calculator. Calculate $\theta(t)$ for $t = 0$ to $t = 0.3$ s with $\Delta t = 0.1$ s. **Fill in a table** like the one shown below.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\theta(t)$</th>
<th>$\dot{\theta}(t)$</th>
<th>$\Delta t \ddot{\theta}(t)$</th>
<th>$2\dot{\theta}(t)$</th>
<th>$\theta(t - \Delta t)$</th>
<th>$\theta(t + \Delta t)$</th>
</tr>
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</tr>
</tbody>
</table>

[Note: DO NOT MAKE THE SMALL ANGLE APPROXIMATION IN THE EQUATION OF MOTION Eq. (1.1). The point of this part and the rest of the problem is to determine how accurate is the analytic solution of the equation of motion with the small angle approximation compared to the numerical solution for the exact equation of motion Eq. (1.1).]  

c) Numerical Solution

Using the previous section as your guide, use a spreadsheet program or MatLab to solve the pendulum problem. 0.1 s time steps should give acceptable results. **Plot and hand in a curve** of $\theta(t)$ for $0 \leq t \leq 20$ s. Include on the same plot the curve corresponding to your analytic solution from part (a). **Include also a printout** of the top few rows of your spreadsheet program or MatLab program to show how you solved the problem.

**From your output** what is period of oscillation? How does it compare to the small angle approximation to the period of $T = 2\pi \sqrt{\frac{l}{g}}$?

Note you can check your results using the simulation “galileo” that you used in the previous problem set.
d) Large Initial Angle

Repeat part (c) for the initial conditions $\theta(0) = 2.8$ rad and $\dot{\theta}(0) = 0$ rad/s. In particular determine the period of oscillation from your numerical procedure.