WKB Review

Symmetric Well

Tunnel Splitting \( 2\Delta \)

Two-Level System \[
\begin{pmatrix}
0 & \Delta \\
\Delta & 0
\end{pmatrix}
\]

\( \Delta \sim \text{Overlap of 'left' + 'right' states.} \)

WKB \( \Delta \sim \Delta \sim e^{i\pi/2} \)

\( H\Psi = E\Psi \)

\( (H-E)\Psi = -\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} + (V-E)\Psi = 0 \)

\( \Psi \sim e^{-kx} \), \( \frac{k^2\hbar^2}{2m} = V-E \)

\( k \sim \sqrt{2mVx/\hbar} \)

Overlap \( e^{-\sqrt{2mVx/\hbar}x} \)
Instantons, Quantum Tunneling, and WKB

Sidney Coleman, The Uses of Instantons, sections 1&2, Berdie lectures, reprinted in Aspects of Symmetry ch. 7.

\[ WKB: \Delta \sim e^{-\sqrt{2mVQ/k}} \]
\[ \Gamma \sim \Delta^2 \]

How to use path integrals? Rotate to imaginary time!

\[ \langle x', t' | x_0, t_0 \rangle = \sum D[x(t)] \ e^{i\frac{\hbar}{\sqrt{2m}} \int x'^2 - V(x) \ dt} \]
\[ r = it; \quad -idr = dt; \quad \dot{x} = (dx/dt)^2 = -(d^2x/dt^2) = x'^2 \]
\[ = \sum D[x(r)] \exp \left[ -\frac{\hbar}{2m} \int x'^2 + V(x) \ dr \right] \]

* Analytic continuation???
* Physics, not math. Gives different information!
* No oscillations! Biggest minimizes Euclidean action:

\[ S_E = \int \frac{1}{2} m x'^2 + V(x) \ dr = \text{Inverted potential} \]

Assume \( V(x) = 0 \) at \( x = \pm a/2 \)
Instanton: Falls in, rolls up other side

Barrier crossing: bounces off turning point

Soliton in time -> Instanton
- Dilute Gas
- Quadratic Fluctuations

How to get WKB?

(1) Variational bound: Ramp

Minimize wrt $\Delta t$: 

$$-\frac{1}{2} m \frac{Q^2}{\Delta t^2} + V = 0$$

$$\Delta t = \sqrt{\frac{1}{2} m Q^2 / V}$$

$$S_E \leq \sqrt{\frac{1}{4} m a^2 V} + \sqrt{\frac{1}{4} m a^2 V} = \sqrt{2mVQ}$$

$$\Delta = \frac{\epsilon}{\sqrt{2mVQ}} e^{-\sqrt{2mVQ}/\Delta t}$$

$V = \int u(x) \, dx$