The competition between entropy and energy in statistical mechanics is not always balanced enough to give a phase transition at finite temperature. Fluctuations (entropy) typically becomes more important as the dimension gets smaller. The lower critical dimension for a given model is the dimension below which no long-range order is possible.

4.1 Ising Lower Critical Dimension. (Dimension dependence) ³

What is the lower critical dimension of the Ising model? If the total energy $\Delta E$ needed to destroy long-range order is finite as the system size $L$ goes to infinity, and the associated entropy grows with system size, then surely long-range order is possible only at zero temperature.

(a) Ising model in $D$ dimensions. Consider the Ising model in dimension $D$ on a hypercubic lattice of length $L$ on each side. Estimate the energy¹ needed to create a domain wall splitting the system into two equal regions (one spin up, the other spin down). In what dimension will this wall have finite energy as $L \to \infty$? Suggest a bound for the lower critical dimension of the Ising model.

The scaling at the lower critical dimension is often unusual, with quantities diverging in ways different from power laws as the critical temperature $T_c$ is approached.

(b) Correlation length in 1D Ising model. Estimate the number of domain walls at temperature $T$ in the 1D Ising model. How does the correlation length $\xi$ (the distance between domain walls) grow as $T \to T_c$? (Hint: Change variables to $\eta_i = S_i S_{i+1}$, which is $-1$ if there is a domain wall between sites $i$ and $i+1$.) The correlation exponent $\nu$ satisfying $\xi \sim (T - T_c)^{-\nu}$ is $1$, $0.63$, and $1/2$ in dimensions $D = 2$, $3$, and $\geq 4$, respectively. Is there an exponent $\nu$ governing this divergence in one dimension? How does $\nu$ behave as $D \to 1$?

¹Energy, not free energy! Think about $T = 0$. 

Assignment 4: Low dimensional systems
Statistical Physics
Physics 7653
James Sethna; Due Tuesday, November 5
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Reading
Cardy, chapter 6
4.2 **XY Lower Critical Dimension and the Mermin-Wagner Theorem.** (Dimension dependence) ③

Consider a model of continuous unit-length spins (e.g., XY or Heisenberg) on a $D$-dimensional hypercubic lattice of length $L$. Assume a nearest-neighbor ferromagnetic bond energy

$$-J \mathbf{S}_i \cdot \mathbf{S}_j.$$  

(1)

Estimate the energy needed to twist the spins at one boundary $180^\circ$ with respect to the other boundary (the energy difference between periodic and antiperiodic boundary conditions along one axis). In what dimension does this energy stay finite in the thermodynamic limit $L \to \infty$? Suggest a bound for the lower critical dimension for the emergence of continuous broken symmetries in models of this type.

Note that your argument produces only one thick domain wall (unlike the Ising model, where the domain wall can be placed in a variety of places). If in the lower critical dimension its energy is fixed as $L \to \infty$ at a value large compared to $k_B T$, one could imagine most of the time the order might maintain itself across the system. The actual behavior of the XY model in its lower critical dimension is subtle.

On the one hand, there cannot be long-range order. This can be seen convincingly, but not rigorously, by estimating the effects of fluctuations at finite temperature on the order parameter, within linear response. Pierre Hohenberg, David Mermin and Herbert Wagner proved it rigorously (including nonlinear effects) using an inequality due to Bogoliubov. One should note, though, that the way this theorem is usually quoted ("continuous symmetries cannot be spontaneously broken at finite temperatures in one and two dimensions") is too general. In particular, for two-dimensional crystals one has long-range order in the crystalline orientations, although one does not have long-range broken translational order.

On the other hand, the XY model does have a phase transition in its lower critical dimension at a temperature $T_c > 0$. The high-temperature phase is a traditional paramagmatic phase, with exponentially decaying correlations between orientations as the distance increases. The low-temperature phase indeed lacks long-range order, but it does have a stiffness – twisting the system (as in your calculation above) by $180^\circ$ costs a free energy that goes to a constant as $L \to \infty$. In this stiff phase the spin-spin correlations die away not exponentially, but as a power law.

The corresponding Kosterlitz–Thouless phase transition has subtle, fascinating scaling properties. Interestingly, the defect that destroys the stiffness (a vortex) in the Kosterlitz–Thouless transition does not have finite energy as the system size $L$ gets large. We shall see that its energy grows $\sim \log L$, while its entropy grows $\sim T \log L$, so entropy wins over energy as the temperature rises, even though the latter is infinite.
4.3 **Long-range Ising.** (Dimension dependence) ③

The one-dimensional Ising model can have a finite-temperature transition if we give each spin an interaction with distant spins.

**Long-range forces in the 1d Ising model.** Consider an Ising model in one dimension, with long-range ferromagnetic bonds

$$\mathcal{H} = \sum_{i>j} \frac{J}{|i-j|^\sigma} S_i S_j. \quad (2)$$

For what values of $\sigma$ will a domain wall between up and down spins have finite energy? Suggest a bound for the ‘lower critical power law’ for this long-range one-dimensional Ising model, below which a ferromagnetic state is only possible when the temperature is zero. (Hint: Approximate the sum by a double integral. Avoid $i = j$.)

The long-range 1D Ising model at the lower critical power law has a transition that is closely related to the Kosterlitz-Thouless transition. It is in the same universality class as the famous (but obscure) Kondo problem in quantum phase transitions. And it is less complicated to think about and less complicated to calculate with than either of these other two cases.