Scaling and power laws are found not only at continuous thermodynamic phase transitions, but also in dynamical systems, earthquakes and avalanches, and even random walks.

1.1 The Gutenberg Richter law. (Scaling) ③

![Fig. 1 Gutenberg Richter Law](image)

The number of earthquakes in 1995 as a function of their magnitude $M$ or their size (energy radiated) $S$.

Power laws often arise at continuous transitions in non-equilibrium extended systems, particularly when disorder is important. We don’t yet have a complete understanding of earthquakes, but they seem clearly related to the transition between pinned and sliding faults as the tectonic plates slide over one another.

The size $S$ of an earthquake (the energy radiated, shown in the upper axis of Figure 1) is traditionally measured in terms of a ‘magnitude’ $M \propto \log S$ (lower axis). The Gutenberg-Richter law tells us that the number of earthquakes of magnitude $M \propto \log S$ goes down as their size $S$ increases. Figure 1 shows that the number of avalanches of magnitude between $M$ and $M + 1$ is proportional to $S^{-B}$ with $B \approx 2/3$. However, it is traditional in the physics community to consider the probability density $P(S)$ of having an avalanche of size $S$.

If $P(S) \sim S^{-\tau}$, give a formula for $\tau$ in terms of the Gutenberg-Richter exponent $B$. (Hint: The bins in the histogram have different ranges of size $S$. Use $P(M)\,dM = P(S)\,dS$.)
1.2 Random Walks. (Scaling) ③

Self-similar behavior also emerges without proximity to any obvious transition. One might say that some phases naturally have self-similarity and power laws. Mathematicians have a technical term generic which roughly translates to ‘without tuning a parameter to a special value’, and so this is termed generic scale invariance.

The simplest example of generic scale invariance is that of a random walk. Figure 2 shows that a random walk appears statistically self-similar.

Fig. 2 Random walk scaling. Each box shows the first quarter of the random walk in the previous box. While each figure looks different in detail, they are statistically self-similar. That is, an ensemble of medium-length random walks would be indistinguishable from an ensemble of suitably rescaled long random walks.

Let \( X(T) = \sum_{t=1}^{T} \xi_t \) be a random walk of length \( T \), where \( \xi_t \) are independent random variables chosen from a distribution of mean zero and finite standard deviation. Derive the exponent \( \nu \) governing the growth of the root-mean-square end-to-end distance \( d(T) = \sqrt{\langle (X(T) - X(0))^2 \rangle} \) with \( T \). Explain the connection between this and the formula from freshman lab courses for the way the standard deviation of the mean scales with the number of measurements.

1.3 Period Doubling. (Scaling) ③

Most of you will be familiar with the period doubling route to chaos, and the bifurcation diagram shown below. (See also Sethna Section 12.3.3).
Fig. 3 Scaling in the period doubling bifurcation diagram. Shown are the points \( x \) on the attractor (vertical) as a function of the control parameter \( \mu \) (horizontal), for the logistic map \( f(x) = 4\mu x(1 - x) \), near the transition to chaos.

The self-similarity here is not in space, but in time. It is discrete instead of continuous; the behavior is the similar if one rescales time by a factor of two, but not by a factor \( 1 + \epsilon \). Hence instead of power laws we find a discrete self-similarity as we approach the critical point \( \mu_\infty \).

From the diagram shown, roughly estimate the values of the Feigenbaum numbers \( \delta \) (governing the rescaling of \( \mu - \mu_\infty \)) and \( \alpha \) (governing the rescaling of \( x - x_p \), where \( x_p = 1/2 \) is the peak of the logistic map). If each rescaling shown doubles the period \( T \) of the map, and \( T \) grows as \( T \sim (\mu_\infty - \mu)^{-\zeta} \) near the onset of chaos, write \( \zeta \) in terms of \( \alpha \) and \( \delta \). If \( \xi \) is the smallest typical length scale of the attractor, and we define \( \xi \sim (\mu_\infty - \mu)^{-\nu} \) (as is traditional at thermodynamic phase transitions), what is \( \nu \) in terms of \( \alpha \) and \( \delta \)? (Hint: be sure to check the signs.)
1.4 Hysteresis and Barkhausen Noise. (Scaling)

Fig. 4 Avalanche size distribution, as a function of the disorder, for a model of hysteresis. The thin lines are the prediction of the scaling form eqn (1), fit to data near $R_c$. The inset shows a scaling collapse; all the data collapses onto the scaling function $D(S^\sigma r)$. (The inset uses the notation of the original paper: the probability $D(S, R)$ is called $D_{\text{int}}$ because it is ‘integrated’ over magnetic field, $\bar{\tau}$ is called $\tau + \sigma \beta \delta$ and $r = (R_c - R)$).

Hysteresis is associated with abrupt phase transitions. Supercooling and superheating are examples (as temperature crosses $T_c$). Magnetic recording, the classic place where hysteresis is studied, is also governed by an abrupt phase transition – here the hysteresis in the magnetization, as the external field $H$ is increased (to magnetize the system) and then decreased again to zero. Magnetic hysteresis is characterized by crackling (Barkhausen) electromagnetic noise. This noise is due to avalanches of spins flipping as the magnetic interfaces jerkily are pushed past defects by the external field (much like earthquake faults jerkily responding to the stresses from the tectonic plates). It is interesting that when dirt is added to this abrupt magnetic transition, it exhibits the power-law scaling characteristic of continuous transitions.

Our model of magnetic hysteresis (unlike the experiments) has avalanches and scaling only at a special critical value of the disorder $R_c \sim 2.16$ (Figure 4). The probability distribution $D(S, R)$ has a power law $D(S, R_c) \propto S^{-\bar{\tau}}$ at the critical point, but away from the critical point takes the scaling form

$$D(S, R) \propto S^{-\bar{\tau}} D(S^\sigma (R - R_c)).$$

Note from eqn (1) that at the critical disorder $R = R_c$ the distribution of avalanche sizes is a power law $D(S, R_c) = S^{-\bar{\tau}}$. The scaling form controls how this power law is altered as $R$ moves away from the critical point. From Figure 4 we see that the main effect of moving above $R_c$ is to cut off the largest avalanches at a typical largest size $S_{\max}(R)$, and another important effect is to form a ‘bulge’ of extra avalanches just below the cut–off.

Using the scaling form from eqn 1, with what exponent does $S_{\max}$ diverge as $r = (R_c - R) \to 0$? (Hint: At what size $S$ is $D(S, R)$, say, one millionth of $S^{-\bar{\tau}}$?) Given $\bar{\tau} \approx 2.03$,
how does the mean $\langle S \rangle$ and the mean-square $\langle S^2 \rangle$ avalanche size scale with $r = (R_c - R)$? (Hint: Your integral for the moments should have a lower cutoff $S_0$, the smallest possible avalanche, but no upper cutoff, since that is provided by the scaling function $D$. Assume $D(0) > 0$. Change variables to $Y = S^\sigma r$. Which moments diverge?)

1.5 Group Projects: Clustering. (Group) ③

In this class, we shall have group projects, where between two and four of you will pick a topic, study it, and present it to the class. Last time I taught this course, two of the topics later became papers (one on mosh pits, one on zombie outbreaks). More traditional topics include

(a) Kosterlitz-Thouless and the lower critical dimension
(b) Flocking and nonlinear hydrodynamics
(c) Disordered systems and dangerous irrelevant variables
(d) Fermi liquid theory and the renormalization group
(e) Quantum phase transitions
(f) Conformal field theory and stochastic Loewner evolution
(g) Jamming
(h) Glasses and models of glasses
(i) Asymptotic analysis of partial differential equations using the RG
(j) Networks: small-world, random, grown, and scale-free
(k) Chaos and turbulence
(l) Depinning transitions
(m) Surface growth
(n) Wetting
(o) Spin glasses

or even non-critical applications of statistical mechanics:

(a) Bioinformatics
(b) Sloppiness in nonlinear least-squares fits
(c) Genomic reconstruction of the tree of life
(d) Statistics and statistical mechanics
(e) Martin-Siggia-Rose techniques
(f) Econophysics
(g) Jarzynski equalities
(h) ...
Begin looking over and discussing topics, and forming into groups. For a few of the topics, more details can be found on the course Web homework site (http://pages.physics.cornell.edu/sethna/teaching/653/HW.html).