2D Turbulence

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Overview

1. Why 2D turbulence?
2. 2D turbulence: the inverse cascade of energy to long length-scales
3. A brief review of fluid dynamics
4. Band-limited forcing and power laws
5. Energy transfer in 2D versus 3D turbulence
6. Applying RG methods to 2D turbulence?
7. Stochastic forcing of 2D N-S
8. A nice way to conclude our discussion
9. Acknowledgments
Approximate 2D systems: atmospheric dynamics

Turbulence on the sphere

Assume surface effects dominate

Weather prediction

Figure: Earth’s atmosphere

Source: The New Republic (website)
2D turbulence: the inverse cascade of energy to long length-scales

Bubbles

Immiscible liquids

Figure: Turbulence on a soap film

Source: Fotolog website
Tractable problem: conformal solution by Polyakov

Shown to exist for every case

Doesn’t mean it’s easy to understand...
Tractable problem: allegorical/numerical solution by Onsager, Kraichnan

Problem related to solvable ones: lattice/vortex gas

Can get scaling and statistics
Equilibrium solution maximizes entropy

Quite different from standard stat physics (more on this soon)
2D turbulence: the inverse cascade of energy to long length-scales

\[ \langle \text{vor\_nf\_early.mov} \rangle \quad \langle \text{psi\_nf\_early.mov} \rangle \]
2D turbulence: the inverse cascade of energy to long length-scales

\( \langle \psi_{nf\text{ late}}.\text{mov} \rangle \)
A brief review of fluid dynamics

- **2D Navier-Stokes:**
  \[
  \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}
  \]

- **Incompressibility:**
  \[
  \nabla \cdot \mathbf{u} = 0
  \]
A brief review of fluid dynamics

2D Navier-Stokes and incompressibility

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} \quad \nabla \cdot \mathbf{u} = 0 \]

- In 2D, the vorticity is a scalar (in the z-direction):
  \[ \omega = (\nabla \times \mathbf{u}) \cdot \hat{z} \]

- For 2D only, we can define the \textit{stream function} \( \psi \), where:
  \[ \mathbf{u} = -\hat{z} \times \nabla \psi \]
  \[ \omega = -\nabla^2 \psi \]
A brief review of fluid dynamics

Fourier transformed 2D vorticity equation

\[
\frac{\partial \omega_k}{\partial t} + \sum_{k'} M_{k,k'} \omega_{k'} \omega_{k-k'} = \nu k^2 \omega_k \quad k \cdot u(k, t) = 0
\]

- \(M_{k,k'}\) takes care of the nonlinear term:

\[
M_{k,k'} = \hat{z} \cdot (k \times k') \left( \frac{1}{k'} - \frac{1}{k} \right)
\]

- Nonlinear advection term responsible for energy transfer to different length-scales

- Energy transfer \(T(k, q, p)\) occurs between a triad of wave numbers:

\[
k \rightarrow k \quad k' \rightarrow q \quad k - k' \rightarrow p
\]
Infinite number of constants of motion of the form:

\[ \oint u \cdot dl = \int \omega \cdot dS = \text{const} \]

The only other invariants are quadratic in \( \omega \):

Conservation of energy

\[
\frac{\partial E}{\partial t} = \frac{\partial}{\partial t} \langle |u_k|^2 \rangle = \langle \frac{\omega_k^2}{k^2} \rangle = 0
\]

Conservation of enstrophy

\[
\frac{\partial}{\partial t} \Omega = \frac{\partial}{\partial t} \langle k^2 |u_k|^2 \rangle = \frac{\partial}{\partial t} \langle \omega_k^2 \rangle = 0
\]
Power laws in the band-limited forcing (BLF) energy spectrum

**Figure:** Long-time energy spectrum of 2D N-S with BLF

**Figure:** Energy pileup at small wavenumbers due to finite grid size
The inverse cascade as seen from the BLF energy spectrum

Inverse cascade?
The energy "cascades" or "is pumped" from shorter to longer length scales

\[ \langle \text{bl\_spec\_256.mov} \rangle \]

- This phenomenon is due solely to the nonlinear term in N-S
- If dissipative (viscous) forces dominate (the nonlinear term):
  - There will be no cascade
  - The power law will be flat for \( k < k_{\text{forcing}} \)
Features of 2D turbulence

- Locality of energy transfer
  
  Small eddies swept along by large eddies

  Similar-sized eddies coalesce

- (Board algebra to introduce 2D energy transfer)
Comparison with features of 3D turbulence

- 3D vortex stretching
- Kolmogorov energy cascade
- Equilibrium (direct cascade)

  - Energy goes to high wavenumbers
  - Viscosity damps out high wavenumbers
  - For numerical solutions, grid cell size can act as viscous cutoff
(More board work to reproduce energy and enstrophy cascade power laws)

2D gives "inverse energy cascade" and "direct enstrophy cascade"

→ energy cascades to small wavenumbers

→ enstrophy cascades to large wavenumbers
Power law prediction through dimensional analysis

\[ E(k) \]

\[ k^{-5/3} \]

\[ k^{-3} \]

\[ E(k) \]

\[ k^{-5/3} \]

\[ k^{-3} \]
Onsager/Kraichnan’s point vortex gas

Considers point vortices interacting with near neighbors

End up with vortex segregation with disorder/"temperature" coming from forcing
Red and green regions correspond to oppositely-signed point vortices

Left snapshot at $t = 0$ with checkerboard initial condition

Right shows like-signed point vortices clustering at $t = 50$

Source: Joyce and Montgomery
Stochastic forcing of 2D N-S

FT 2D stream function equation with stochastic forcing

\[ \dot{\psi}_k + \sum_{k'} \tilde{M}_{k,k'} \psi_{k'} \psi_{k-k'} + \nu k^2 \psi_k = W_k(t) \]

The stochastic (white noise) term \( W_k \) has:

- an amplitude selected at random from a time-scaled Gaussian distribution at each time step

- a randomly-selected phase at each time step

\[ \rightarrow \text{Scaled random walk in time applied at each } k \text{ (Brownian motion)} \]
To get the stochastic term $W_k$:

1. Using theory of hydrodynamic fluctuations, consider fluctuations in pressure tensor $\delta P$.

2. $\delta P$ turns out to be anti-symmetric.

3. Taking curl of N-S (to get vorticity equation) gives white noise fluctuation term: $\nabla^2 W$.

4. $\omega = -\nabla^2 \psi \rightarrow$ only have $W$ in equation of motion for stream function.
\langle \text{vor}_\text{wn}.\text{mov} \rangle \quad \langle \text{psi}_\text{wn}.\text{mov} \rangle

- Ising-model-like scale invariance

- Like-signed vortices cluster $\rightarrow$ correlation with neighbors

- Striking similarities to MD simulations of Onsager’s discrete vortex model

- Thermal fluctuations give rise to discrete vortices on dissipation scale?
Energy spectrum for white noise forced fluid

Figure: Energy spectrum shows evidence of $k^{-5/3}$ power law
Miller and Cross and a thermodynamic explanation of the inverse cascade

Final state of 2D turbulent system should maximize entropy

Clustering of local regions of vorticity maximizes number of microstates
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Two-dimensional Turbulence