If there is a phase transition there is very likely a
connection to stat mech. Why are some problems "hard"
& some "easy."

Consider the NPP Cost = \[ \sum_{j=1}^{N} a_j s_j \] $\equiv$ "Energy"

Minimizing $E \Rightarrow$ equivalent to minimizing \( \left( \sum_{j=1}^{N} a_j s_j \right)^2 = \sum_{j=1}^{N} a_j a_j s_j \)

viz equivalent to finding ground state of AFM Ising model with infinite

gauge and with coupling $J_{ij} = a_i a_j$ (model for spin glasses)

Let us try understanding the "physics" of this model. Consider a set of numbers $a_j$.

Partition function $Z = \sum_{\{s_j\}} e^{-\frac{1}{T} \sum_{j=1}^{N} a_j s_j}$

The sum can be done by introduction of delta function

\[
Z = \sum_{\{s_j\}} \int e^{-|x|} \delta(x - \sum_{j=1}^{N} a_j s_j) \, dx
\]

Use $\delta(x-a) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-a)}$

\[
Z = \sum_{\{s_j\}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, e^{ikx} e^{-ik \sum_{j=1}^{N} a_j s_j} \, dx
\]

\[
= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx \, e^{-|x|} e^{ikx} \left( \sum_{s_1 = \pm 1} e^{-ik a_1 s_1} \right) \left( \sum_{s_2 = \pm 1} e^{-ik a_2 s_2} \right) \ldots
\]

\[
= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx \, e^{-|x| + ikx} \prod_{j=1}^{N} \frac{2 \cos(a_j k)}{1 + k^2}
\]
just make things equally weighted

\[ k = \tan y \]

\[ dk = \sec^2 y \, dy \quad \Rightarrow \quad \frac{dk}{1+k^2} = dy \]

\[ 1+k^2 = \sec^2 y \]

\[ Z = 2^N \int_{-\pi/2}^{\pi/2} d\theta y \frac{N}{\pi} \cos \left( a_i \tan \frac{\theta}{\pi} \right) \]

\[ \exp \left( \frac{N}{\pi} \sum_{i=1}^{N} \ln \left( \cos a_i \right) \right) \frac{1}{N} \]

\[ e^{NG(y)} \]

\[ Z = 2^N \int_{-\pi/2}^{\pi/2} d\theta y \, e^{NG(y)} = \int_{-\pi/2}^{\pi/2} Z(y) \, dy \]

\[ G(y) = \frac{1}{N} \sum_{i=1}^{N} \ln \left( \cos a_i \right) \]

Bring up: the question - we've just considered ONE particular set of \( a_i \).

But we're more interested in what a typical behaviour would be like - i.e.
we can consider some sort of average or typical partition function.

Would it be wise to average \( \langle Z \rangle \)? i.e. \( \exp \langle \ln Z \rangle \) - the
more physical quantity (since \( F \sim -T \ln Z \)) i.e. equivalently do we average
\( e^{NG(y)} \) - a wildly varying function or \( e^{N\langle G(y) \rangle} \) \( G(y) \) is logarithmic
& super from minor fluctuations. This is the difference between annealed
\( \langle Z \rangle \) and \( \exp \langle \ln Z \rangle \) (free energy).

\[ Z = 2^N \int_{-\pi/2}^{\pi/2} d\theta y \frac{1}{\pi} e^{NG(y)} \]

\[ G(y) \sim \langle \ln \cos \left( a_i \tan \frac{\theta}{\pi} \right) \rangle \]
Task is to compute abut most dominant contributions to the integral come from region where \( G(y) \) is a maxima.

Around each maxima\( G(y) \approx G(y_0) - \frac{1}{2} \left| G''(y_0) \right| (y - y_0)^2 \) [Laplace's method]

(Note first derivative vanishes)

\[
\int e^{NG(y)} \, dy \approx e^{NG(y_0)} \int e^{-NG''(y_0) \frac{(y - y_0)^2}{2}} \, dy \\
\sim e^{NG(y_0)} \sqrt{\frac{2\pi}{G''(y_0)}}
\]

One can compute \( G'(y) = \left\langle \frac{a}{T} \tan \left( \frac{a}{T} \tan y \right) \right\rangle \sec^2 y = 0 \) \( \delta y \)

\[ \Rightarrow y_k = \tan^{-1} \left( \frac{\pi k}{T} \right) \quad k = 0, \pm 1, \pm 2, \ldots \]

Turns out that \( G(y_k) = 0 \)

\[ G''(y_k) = \left\langle a^2 \right\rangle \left( 1 + \left( \frac{\pi k}{T} \right)^2 \right)^{-2} \]

\[ Z = 2^N \sqrt{\frac{2}{\pi N}} \sum_k \frac{1}{\sqrt{G''(y_k)}} \]

\[ = \left( \frac{T}{\left\langle a^2 \right\rangle} \right)^{\frac{N}{2}} \sum_k \frac{1}{1 + \left( \frac{\pi k}{T} \right)^2 k^2} \]

\[ = \frac{2^m}{\sqrt{3}} \quad \text{can be exactly summed} \]

\[ Z = \left( \frac{1}{\sqrt{\frac{\pi}{6}}} \right) \left( \frac{2^{N-M}}{\sqrt{N\pi}} \right) \left( \coth \left( \frac{1}{T} \right) \right) \]

\[ F(T) = -T \ln Z = T(M-N) \ln 2 + \frac{T}{2} \ln \frac{N \pi}{6} - T \ln \coth \left( \frac{1}{T} \right) \]

\[ = TN(\kappa - \kappa_c) \ln 2 - T \ln \coth \left( \frac{1}{T} \right) \]
\[<E>_T = \frac{1}{\sinh \frac{1}{T} + \cosh \frac{1}{T}} = -\frac{3 \ln 2}{2p} \]

\[S = N (k_c - k) \ln 2 + \tilde{S}(T) = -\frac{\partial F}{\partial T} \]

\[K_c = 1 - \frac{\ln \frac{T/6}{N}}{2N \ln 2} \]

\[K = \frac{M}{N} \]

\[\tilde{S}(T) = \ln \coth \left(\frac{1}{T}\right) + \frac{1}{T} \frac{\coth^2 \left(\frac{1}{T}\right) - 1}{\coth \left(\frac{1}{T}\right)} \]

\[\tilde{S}(T) \ (\text{for large } T) \sim \ln T \]

\[\tilde{S}(\text{small } T \to 0) \sim 0 \]

For \( S \) to be large & "extensive"

\[K_c > K \Rightarrow \frac{M}{N} < 1 \]

\[<E>_T \to 0 \quad \text{as } T \to 0 \]

\[\frac{e^{1/T} + e^{-1/T}}{e^{1/T} - e^{-1/T}} \]

\[\frac{e^x}{x} \left(\frac{e^{2x} + 1}{e^{2x} - 1}\right) \]

\[\frac{4e^{2x}}{(e^{2x} + 1)(e^{2x} - 1)} \times e^{-2x} \]

We have a large number of states which give 0 partition cost

Consider regime \( K > k_c \) then \( S \) appears to be large & negative

We can reject this on physical grounds (and somehow claim it is an artifact of our approximations - I'm waving my hand here)
We can see a finite temperature the system can go to allow

\[ \ln T_0 + N (K_c - K) \ln 2 = \ln 2 \]

\[ T_0 = 2^N (K_c - K) = 2 \]

\[ T_0 = 2^{NK_c - N + \frac{1}{2} \log_2 N} \sqrt{\frac{2\pi}{3}} \]

\[ \langle E \rangle_{T_0} \rightarrow \frac{1}{\sinh \left( \frac{1}{T_0} \right) \cosh \left( \frac{1}{T_0} \right)} \sim T_0 \sim 2^{N_0 - N + \frac{1}{2} \log_2 N} \sqrt{\frac{2\pi}{3}} \]

\[ S = \ln 2 \]

I'd plotted this prediction & it agrees! (Hence stat mech has an answer for this observation albeit a bit handwavy - can be made rigorous hopefully!)

For \[ N=45 \quad 20 \quad K_c \approx 0.9 \]
\[ 1 - \frac{\ln(\frac{\pi x 20}{6})}{2 \times 20 \times \ln 2} \approx 0.91529 \]
Optimal cost and theoretical prediction, $N=20$