I. 1D ISING MODEL WITH INVERSE SQUARE INTERACTIONS

A. Hamiltonian

We consider a 1D chain of spins with inverse square interactions.

\[-\beta H = K \sum_{i,j \neq j} S_i S_j (i - j)^2\]  

where \(S_{i,j} = \mp 1\) and for a ferromagnetic system \(K > 0\). The reason that makes this system interesting is not that it corresponds to a real world physical system. It is that the renormalization group flow equations of this system are extremely similar to that of a Kosterlitz Thouless transition, and hence it provides a playground to study this transition.

B. Excitation energy of a single domain wall

We now calculate the energy required to create a single domain wall (a kink) in a ferromagnetic chain of length \(L\).

\[-\beta \Delta E = \int_{-L/2}^{0} dr \int_{0}^{L/2} dr' \frac{-2K}{(r - r')^2} = -2K \log \frac{L}{a}\]  

We have introduced a low distance cut-off \(a\) which is equal to the lattice spacing (the distance between two neighboring spins).

C. Interaction between kinks

We now aim to write the hamiltonian (1) which shows the interaction between spins, in terms of positions of domain walls. This way one should be able to manifest the interactions between walls.

Let us denote the positions of kinks as \(q_m\) such that the spins between and including \((q_m - 1)\) and \((q_m + 1)\) have the same sign. Then we can write the double summation in terms of spins as

\[\sum_{i,j} \rightarrow \sum_{m,n} \sum_{i=q_m}^{q_{m+1}-1} \sum_{j=q_n}^{q_{n+1}-1} \]  

where the indices \(m,n\) run over the kinks whereas \(i,j\) run over the spins. Using this fact, we can write the hamiltonian (1) as

\[-\beta H = \frac{K}{2} \sum_{m,n} \sum_{i=q_m}^{q_{m+1}-1} \sum_{j=q_n}^{q_{n+1}-1} (-1)^{m+n} (i - j)^2 - \sum_{m} \mu.\]  

We have used the fact that

\[S_i S_j = \begin{cases} +1, & \text{if } m \text{ and } n \text{ are both even or both odd.} \\ -1, & \text{if one of } m \text{ and } n \text{ is odd and the other one is even.} \end{cases}\]  

so that \(S_i S_j = (-1)^{m+n}\) where \(i\) is in \(m\)th cluster and \(j\) is in \(n\)th cluster. Also note that our new summations do not exclude the \(i = j, m = n\) term so we need to subtract a (possibly divergent) term for each kink. The chemical potential \(\mu\) is introduced to serve this aim. The value of it depends on the short range behavior of the interaction which is probably not universal. \(\mu\) corresponds to the core energy \(E_c\) of vortices discussed before. (—give a reference??)

We now convert the summations over the spin indices to integrals. We have the liberty to treat the low distance cutoffs poorly in our integrals since the exact nature of \(\mu\) is not specified. We get

\[-\beta H = \frac{K}{2} \int_{q_m}^{q_{m+1}} dr \int_{q_n}^{q_{n+1}} \frac{1}{(r - r')^2} = N \mu\]  

In the thermodynamic limit \(L \rightarrow \infty\) the position of the kink is unimportant and converting the summations to integrals we get:

\[-\beta \Delta E = \int_{-L/2}^{0} dr \int_{0}^{L/2} dr' \frac{-2K}{(r - r')^2} = -2K \log \frac{L}{a}\]
where $N$ is the total number of kinks. An inverse square term integrated twice will clearly give rise to a logarithmic interaction. After taking the integrals and playing with couple of dummy variables we get

$$-eta H = 2K \sum_{m,n} (-1)^{m+n} \log |q_m - q_n| - N\mu. \quad (7)$$

It is now clearly seen that domain walls have an interaction proportional to the logarithm of the distance between them. Also, in a ferromagnetic system (when $K > 0$) kinks of opposite signs repel each other.

One other point to note is that the hamiltonian of kinks have the factor $2K$, whereas the hamiltonian for spins had $K/2$. So, we can say that $K_{\text{kink}} = 4K_{\text{spin}}$.

### D. When are domain walls favorable?

Creation of a kink in a ferromagnetic system has an energy cost, however, it increases entropy and hence it might be favorable to have a domain wall at finite $T$. We now make a heuristic argument to see when this is the case.

We have previously shown (in 1B) that the energy cost of having a domain wall is

$$\beta H = 2K \log \frac{L}{a}. \quad (8)$$

The entropy of such a configuration can be calculated as follows: By definition entropy is $S = k_B \log \#$, where $k_B$ is the Boltzmann constant which is taken to be equal to one from now on and $\#$ is the total number of possible configurations consistent with the macroscopic state. A chain of length $L$ has $L/a$ lattice sites, so the kink can be at $L/a$ different positions. Therefore $\# = L/a$ and so $S \sim \log L/a$.

Putting these together, the free energy $F = E - TS$ of the system is

$$F \sim (2K - 1) \log \frac{L}{a}. \quad (9)$$

(Note that the temperature in $\beta$ has canceled the $T$ in front of the entropy term.) In the thermodynamic limit $\log L/a \to \infty$ and the free energy gain $F$ is either positive infinity of negative infinity depending on the value of $K$. We see that the critical value of $K$ is

$$K_c = \frac{1}{2} \quad (10)$$

and if $K > K_c$ domain walls are not favorable whereas if $K < K_c$ domain walls are (infinitely) favorable.

Although this discussion have considered creation of a single domain wall, it can be generalized to include more than one kinks as well.

### E. Handwavy RG Calculation - 1

We now study the behavior of this 1D Ising system under a renormalization of length scales. Namely, we change the low distance cutoff $a$ and calculate the corresponding change in the parameters in the hamiltonian.

The hamiltonian has two parameters, the interaction strength $K$ and the chemical potential $\mu$. For reasons of simplicity, we define a fugacity

$$y = e^{-\mu} \quad (11)$$

and from now on study the behavior this parameter instead of $\mu$.

The partition function $Z$ is defined as

$$Z = \sum e^{-\beta H} \quad (12)$$

where the sum is over all possible configurations of the system. We consider a state with a single kink. As discussed before, the kink can be at $L/a$ different positions each of which have the same energy. So,

$$Z \approx y^{L/a} e^{-2K \log L/a} \quad (13)$$

$$\propto y^{a^{-1+2K}}. \quad (14)$$

Now, we make our RG transformation by letting $a \to ae^l$, where $l \ll 1$ is a dimensionless number. The renormalized partition function $Z'$ will have the same form but the factors of $a$ in it will be multiplied by $e^l$. So,

$$Z' \propto y(ae^l)^{-1+2K} \quad (15)$$

We see that in order to have $Z' = Z$, it is sufficient to renormalize $y$ such that

$$y \to ye^{l(1-2K)}. \quad (16)$$

This gives our first flow equation:

$$\frac{dy}{dl} = y(1 - 2K) \quad (17)$$

### F. Handwavy RG Calculation - 2

We now turn our attention to the other parameter in the hamiltonian, that is, $K$. But before doing that, we will consider the physical meaning of renormalization first.
We define a blip as two consecutive kinks separated by a distance $a$. That is, a blip corresponds to one single spin rotated in the opposite direction in a ferromagnetic state. Considering the kinks as charges, we can say that a blip is two opposite charges separated by the low distance cutoff. So, it correspond to a dipole in electrostatics. Also, an analogy between them and the vortex pairs in 2D superfluid helium can be constructed.

When we make the renormalization group transformation $a \rightarrow ae^l$, blips are washed away since their widths are less than the new cutoff $ae^l$. But their effect will be present in the new values of parameters in the hamiltonian. In electrostatics, free charges’ screening change the power law of the interaction, whereas free dipoles screen in a different manner. Namely, free dipoles change the interaction strength $\epsilon$, which is the permittivity of the medium. Similarly, we expect the blips to renormalize the interaction strength $K$. In order to calculate the amount of this change, we consider the interaction of a kink with two consecutive kinks. The reason why we choose this specific case might seem unclear to the reader, and the authors share the confusion. A full formal calculation (—cite Cardy??) shows that this is the simplest system that can be used to study the renormalization. We do not seek to justify this choice and just carry on our handwavy argument.

There might be an arbitrary number of blips between the two kinks before renormalization. After renormalization, all of these blips will be washed away. So the partition function $Z'$ is a sum over the partition functions of all configurations with different number of blips:

$$Z' = \sum_{n=0}^{\infty} \left( \int_{a}^{ae^l} da \right)^n Z_n$$

where $Z_n$ is the partition function of a configuration with $n$ blips. Each $Z_n$ is to be integrated over $a n$ times since each blip can have a width between $a$ and $ae^l$. We hope that the sloppiness of notation in equation 18 is forgivable.

Now, let us calculate the terms $Z_n$ and see if we can get a reasonably simple closed form for them. We do not include the fugacity corresponding to the kinks and the interaction between the consecutive kinks in our partition functions, since they will cancel anyway. $Z_0$, which is the partition function corresponding to no blips has the form:

$$Z_0 = e^{+4K \log \frac{1}{e^{a/2}} - \pi}$$

and it is not to be integrated over $a$, since there are no blips. Now, we consider the case with a single blip in order to calculate $Z_1$. We denote the position of the center of this blip with respect to the kink on its left as $x$. $Z_1$ includes an integral over $x$, which correspond to the summation over all possible positions of the blip. The partition function including only the terms relevant
to our discussion takes the form
\[ Z_1 = \int_{x=a}^{R-a} e^{-\beta H} \frac{dx}{a} \]
\[ = \frac{y}{\pi} \int \exp \left(4K \left[ \log L - \log (L + R) - \log (L + x - a/2) + \log (L + x + a/2) \right] \right) \frac{dx}{a} \]
\[ = \frac{y}{\pi} e^{4K \log \frac{L}{L + R}} \left( 1 + \frac{4Ka}{L + x} \right) \frac{dx}{a} \]
The term 1 in the integrand will just add a constant to the energy and can be omitted. Thus, after the integration \( Z_1 \) turns out to be
\[ Z_1 = e^{4K \log \frac{L}{L + R}} \left[ -4Ky^2 \log \frac{L}{L + R} \right]. \] (23)

Now, we integrate this over \( a \) once:
\[ \int_a^{a(l+1)} da Z_1 = e^{4K \log \frac{L}{L + R}} \gamma \] (24)
where we have defined
\[ \gamma = -4Ky^2 a \log \frac{L}{L + R}. \] (25)

Note that the upper limit of the integration was \( a(1+l) \), which is equal to \( a e^{l} \) in the limit \( l \ll 1 \).

In order to calculate \( Z_2 \), the question whether we should consider the interactions between two blips arises. However, it is easy to show that interaction between two blips decay faster than \( 1/r \), and so is short ranged. As one does not expect short range behavior to affect the universal behavior, we can safely ignore this interaction. Denoting the positions of kinks as \( x_1 \) and \( x_2 \), we find
\[ Z_2 = y \int \frac{dx_1 dx_2}{a^2} \exp \left(4K \left[ \log \frac{L}{L + R} - \log (L + x_1 - a/2) + \log (L + x_1 + a/2) \right] \right) \]
The two integrals separate and we find
\[ \int_a^{ae} da_1 \int_a^{ae} da_2 Z_2 = e^{4K \log \frac{L}{L + R}} \gamma^2_{2!} \] (27)
We have inserted the factor of \( 2! \) in order to account for the indistinguishability of blips. (In general, as we did not consider any restrictions on the positions of blips, \( Z_{2a} \) will be overcounted by a factor of \( n! \)) The reasoning can be generalized to any number of blips, and considering the distance between the two kinks \( R \) to be \( R \gg a \), we can consider the maximum number of blips in between be infinite. Hence,
\[ Z' = Z_0 + \int da Z_1 + \int da_1 \int da_2 Z_2 + \ldots \] (28)
\[ = e^{4K \log \frac{L}{L + R}} \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} \] (29)
\[ = \exp \left(4K \log \frac{L}{L + R} \left[ 1 - y^2 a \right] \right) \] (30)
\[ = e^{4K' \log \frac{L}{L + R}} \] (31)

Redefining the fugacity \( y \rightarrow \sqrt{a} y \), we get
\[ K'_{(20)} = K(1 - y^2 l), \] (32)
and therefore
\[ \frac{dK}{dl} = -Ky^2. \] (33)

But this is not the whole story yet. We have showed that a blip affects the interaction of kinks next to it. In general, the interaction between two kinks can be affected by four different blips, one at each side of each kink. So, we multiply our result by 4 to get
\[ \frac{dK}{dl} = -4Ky^2 \] (34)
which is the second of our renormalization group equations.

Although there are many points which are not totally well founded in our proofs, we hope that we gave a intuitively satisfying derivation of the RG flow equations.

II. RENORMALIZATION GROUP FLOW [? ], [? ]

In the previous section, we found the Kosterlitz-Thouless renormalization group equations to be given by
\[ \begin{cases} \frac{dK}{dl} = -4Ky^2, \\ \frac{dy}{dl} = (1 - 2K)y, \end{cases} \] (35)
where \( K = J/k_BT \) is the reduced coupling parameter, and \( y = e^{-\mu/k_BT} \) is the fugacity. From here, we can see that for \( K > 1/2 \), the fugacity decreases with increasing \( l \), i.e. \( y \) is an irrelevant parameter. This regime corresponds to the ferromagnetic phase in our Ising model or, in terms of the vortex model discussed earlier, to vortex pairs existing in closely bound pairs, so that at longer length scales the overall density of vortices becomes small. Similarly, for \( K < 1/2 \), \( y \) becomes a relevant parameter, and the system is in the paramagnetic phase, with domain walls proliferating. In the vortex model, we interpret this as the existence of unbound vortices, the presence of which can be felt even at larger
In the vicinity of curves that flow to the critical point \( (K = \frac{1}{2}, y = 0) \), let us define \( x = 1 - 2K \). In the vicinity of \( (x = 0, y = 0) \), then, the RG equations in (35) become
\[
\begin{aligned}
\frac{dx}{dl} &= 4(1-x)y^2 \approx 4y^2, \\
\frac{dy}{dl} &= xy.
\end{aligned}
\] (36)
Combining these two equations, we can obtain the trajectories of the solutions:
\[
4y\frac{dy}{dl} = x\frac{dx}{dl} \Rightarrow \frac{d}{dl}(x^2 - 4y^2) = 0.
\] (37)
Thus, the trajectories are given by \( x^2 - 4y^2 = C \), for arbitrary constant \( C \). We can now plot the RG flow diagram, with the following points in mind:

- \( y \) is non-negative. The flows are restricted to the upper-half \( (x, y) \) plane.
- For \( C = 0 \), we obtain the lines \( y = \pm \frac{\sqrt{2}}{2} \). By the second relation in Eq. (36), points on the line \( y = -\frac{\sqrt{2}}{2} \) \( (y = +\frac{\sqrt{2}}{2}) \) will flow towards \( y = 0 \) \( (y = \infty) \).
- For \( C > 0 \), the trajectories are hyperbolas below the lines \( y = \pm \frac{\sqrt{2}}{2} \). Again, by the second of the RG equations, the trajectories below the line \( y = -\frac{\sqrt{2}}{2} \) flow towards the fixed line \( y = 0 \), while those below \( y = +\frac{\sqrt{2}}{2} \) flow to \( y = \infty \).
- For \( C < 0 \), the trajectories are hyperbolas in the region above \( y = \pm \frac{\sqrt{2}}{2} \) flowing towards \( x, y = \infty \).

The RG flow diagram is shown in Fig. 1. The particular trajectory \( y = -\frac{\sqrt{2}}{2} \) forms the boundary which separates curves that flow to \( y = 0 \) and curves that approach \( y = \infty \). It is therefore the critical surface between these two distinct phases.

For a system with initial parameters \( (x_0(T), y_0(T)) \), we can plot the evolution of the system as the temperature is varied by noting that:
\[
y = e^{-\mu/\hbar x} = e^{-\mu K/k_B x} \sim e^{\mu x/2k_B J}.
\] (38)
The evolution of a particular system is also shown in Fig. 1. The temperature at which the curve crosses the critical surface marks the critical temperature \( T_C \) of the system. We derive below several important features of the coupling \( K \) and the correlation length \( \xi \) at this critical temperature.

### A. Renormalized \( K \) in low \( T \) phase

Let us examine how the coupling \( K \) renormalizes as the temperature approaches \( T_C \) from the low \( T \) phase. In this region, we have \( C > 0 \), and since \( C \) vanishes at the phase transition, we write it in the form \( C = B^2(T_C - T) \), for some \( B \). This gives a trajectory
\[
x_0^2 - 4y_0^2 = B^2(T_C - T).
\] (39)
After repeated coarse-graining, we know that \( y(l \rightarrow \infty) \rightarrow 0 \), so that the renormalized coupling becomes:
\[
x_R = -B\sqrt{T_C - T} \quad \text{or} \quad K_R = \frac{1}{2} + \frac{B}{2}\sqrt{T_C - T}.
\] (40)
The constant term in the expression for \( K_R \) can be understood as a “universal jump” in the coupling. Previously, we discussed the transition that occurs between the normal and superfluid phases of helium films. We identified \( K \) to be the “stiffness” parameter of a superfluid, which was related to the superfluid mass density \( \rho_S(T) \) by
\[
K = \frac{\hbar^2}{2m^2k_B} \frac{\rho_S(T)}{T}.
\] (41)
Our derivation above shows that at \( T = T_C \), the ratio \( \rho_S(T_C)/T_C \) must be a constant, regardless of the value of \( \rho_S \) at temperatures lower than \( T_C \). This feature can be seen in Fig. 2, where the superfluid mass density of various films is plotted as a function of \( T \). One can also see the square-root dependence of the mass density on \( T \) just below the critical temperature, as predicted by Eq. (40).

### B. Correlation length in high \( T \) phase

We now compute the correlation length \( \xi \) of a system that is slightly above \( T_C \). In this region, we have \( C < 0 \), and we write \( C = -B^2(T - T_C) \). The RG equation for \( x \) then gives
\[
\frac{dx}{dl} = 4y^2 = x^2 + B^2(T - T_C),
\] (42)
FIG. 2: Superfluid mass density $\rho_S(T)$ of helium films of various height, substrate, etc. Note that $\rho_S(T_c)/T_c$, which marks the slope of the straight line, remains constant over different films. Figure obtained from [?].

which we may integrate directly:

$$l = \int_{x_0}^{x(l)} \frac{dx}{x^2 + B^2(T - T_C)}$$

$$= \frac{1}{B\sqrt{T - T_C}} \arctan \left( \frac{x}{B\sqrt{T - T_C}} \right) \bigg|_{x_0}^{x(l)}. \quad (44)$$

We set the upper limit of integration to be $x(l) \approx 1$, since this corresponds to $K \approx 0$, the point at which the domain walls (or vortices) become non-interacting. As we let $T \to T_C$, Eq. (44) reduces to

$$l = \frac{1}{B\sqrt{T - T_C}} \left( \frac{\pi}{2} - \left( \frac{\pi}{2} \right) \right) = \frac{\pi}{B\sqrt{T - T_C}}. \quad (45)$$

The correlation length is therefore given as

$$\xi \sim ae^l = ae^{\frac{\pi}{B\sqrt{T - T_C}}}, \quad (46)$$

where $a$ is the short-distance cutoff of the system. Unlike the systems we have encountered up to this point, the correlation length does not have an algebraic divergence at the critical temperature. Instead, the divergence is much “softer”, and there is only an essential singularity at $T = T_C$. This means that physical quantities that depend on derivatives of $\xi$ will show no particularly drastic behavior at $T_C$. For example, let us consider the singular part of the specific heat $C_{sing} = -T \frac{\partial^2 f_{sing}}{\partial T^2}$. We have learned that the singular part of the free energy scales as $f_{sing} \sim |t|^{d/y_t}$, where $d$ is the dimension of the system, and $y_t$ is the RG eigenvalue corresponding to the temperature parameter $t$. By definition of the critical exponents $\alpha$ and $\nu$, we have that:

$$f_{sing} \sim |t|^{2-\alpha} \sim |t|^{d\nu} \sim \xi^{-d}, \quad (47)$$

where we have used the hyperscaling relation $\alpha = 2 - d\nu$ in the second $\sim$ relation. Thus, the specific heat is

$$C_{sing} = -T \frac{\partial^2}{\partial T^2} \xi^{-2}. \quad (48)$$

Since $\xi$ has only an essential singularity at $T_C$, the specific heat must be finite and smooth at the critical temperature. Fig. 3 shows a plot of the specific heat of a particular XY model. As predicted, there is no special behavior at $T_C$.

FIG. 3: Specific heat of a particular XY model. Note that the curve shows no peculiarity at $T_C$. Figure obtained from [?].