1D Ising Model With $1/2$ Interactions

(Furur Birol, Ben Hunt, Don Teo, James Sethna)

Consider a 1D chain of spins.

$$-\beta \mathcal{H} = \frac{K}{2} \sum_{i \neq j} \frac{S_i S_j}{1-|S_i S_j|^2}$$

$$S_i S_j = \mp 1$$

Ferromagnetic: $K > 0$.

**Excitation energy of a single domain wall (=kink)**

$$-\beta \Delta E = \int_{r'}^{r} \int_{r_{1/2}}^{r} \frac{-2K}{(r-r')^2} = -2K \log \frac{L}{a}$$

($L$ is the system size)

**Interaction between kinks:**

Let kinks be at positions $q_m$ such that spins between including $(q_m)$ and $(q_{m+1} - 1)$ are the same. Then:

$$-\beta \mathcal{H} = \frac{K}{2} \sum_{i \neq j} \frac{S_i S_j}{1-|S_i S_j|^2} = \frac{K}{2} \sum_{m,n} \sum_{i=q_m}^{q_{m+1}} \sum_{j=q_n}^{q_{m+1}} \frac{(-1)^{m+n}}{(i-j)^2} - \sum M$$

$m,n$ summations are over kinks. $m,n \in (-\infty, \infty)$

$i, j$ summation over spins.

$M$ is a chemical potential inserted by hand; because we did not treat $m=n$ terms nicely, $M$ corresponds to the core energy $E_c$ of vortices discussed before.

Note that $S_i S_j = \begin{cases} 1 & \text{if both are in even or odd clusters} \\ -1 & \text{elsewhere} \end{cases}$

So $S_i S_j = (-1)^{m+n}$
Converting sums to integrals (and again treating low distance cutoffs poorly):

\[ - \beta W = \frac{k}{2} \sum_{m,n} (-1)^{m+n} \int_{q_m}^{q_{m+1}} \int_{q_n}^{q_{n+1}} \frac{1}{(r-r')^2} \]  

where \( N = \) number of kinks.

After taking the integrals and changing some dummy variables we get:

\[ - \beta \delta E = 2k \sum_{m,n} (-1)^{m+n} \log |q_m - q_n| - N/M \]

Note that \( K_{\text{link}} = 4K_{\text{spin}} \).

\* When is a domain wall favorable?

Consider a single kink:

\[ \beta E \sim 2k \log \frac{L}{a} \]

\[ S \sim \ln \frac{L}{a} \]

\( L/a \) is the number of positions that the kink can be.

So, free energy is:

\[ F = E - TS \]

\[ \sim (2k-1) \log \frac{L}{a} \]

\[ \left[ K_c = \frac{1}{2} \right] \]

\( K > K_c \) : D. W. not favorable

\( K < K_c \) : D. W. favorable.
(Hand waving) R.G. Calculation

1. Renormalization of $\gamma = e^{-\mu}$
   Define fugacity: $\gamma = e^{-\mu}$.
   Consider the partition function of a single kink:

   \[ Z = \left( \frac{L}{a} \right) e^{-\mu - 2k \log\frac{L}{a}} \]

   \[ \downarrow \]
   \# of positions of kink

   \[ Z \propto \gamma^{-1+2k} \]

   Renormalization: $a \to a e^\ell$ (where $\ell \ll 1$)

   For $Z' = Z$, we need: $\gamma \to \gamma e^{\ell(1-2k)}$

   So: $\frac{d\gamma}{d\ell} = \gamma (1 - 2k)$

   1st R.G. flow equation
Renormalization of $k$

"The blip sum"

A blip is: $\frac{1}{a}$

One can consider $I$ is a charge and $J$ as a positive charge. Then a blip is a dipole, so blips don't screen and change the form of the interaction like free charges do, but they change $k$. (Interaction strength, which is permittivity of the medium.)

Consider the interaction of a kink with two consecutive kinks:

$$I / \ldots / I$$

$$\Rightarrow$$

$$R$$

$$L$$

$$Z' = \int da \left( \sum_{n=0}^{\infty} Z_n \right)$$

[Actually, $\sum_n$ is integrated n times, so this is not totally correct.]
\[
Z_0 = e^{4k \log \frac{L}{L+R}}
\]

\[
Z_1 = \int_{x=0}^{x=R} e^{-\beta_0 x} \, dx
\]

\[
Z_1 = \int_{x=0}^{x=R} e^{-\beta_0 x} \, dx
\]

\[
= y^2 \int_{x=0}^{x=R} e^{-\beta_0 x} \, dx
\]

\[
= y^2 e^{4k \log \frac{L}{L+R}} \int_{x=0}^{x=R} \frac{dx}{e^{4k \frac{L}{L+R}}}
\]

\[
= y^2 e^{4k \log \frac{L}{L+R}} \int_{x=0}^{x=R} \left(1 + 4k \frac{L}{L+R} \right) dx
\]

Add a constant.
Omit it.

\[
= y^2 e^{4k \log \frac{L}{L+R}} \left(-4k y^2 \log \frac{L}{L+R} \right)
\]

\[
\int_{\alpha}^{1+\alpha} d\alpha Z_1 = e^{4k \log \frac{L}{L+R}} \left(-4k y^2 \log \frac{L}{L+R} \right)
\]
\[ z_2 = \frac{x_2}{x_1} \]

Omit interaction between blips as it is even weaker than 1/r.

\[ z_2 = y^n \int \frac{dx_1 dx_2}{a^2} e^{4k \left( \log \frac{L}{2} + \log \frac{L + x_1 - \frac{x_2}{2}}{L + x_2 - \frac{x_1}{2}} \right)} \]

Two two integral separate and:

\[ \int dx, \int dx_2 \quad z_2 = e^{4k \log \frac{L}{2}} \quad y^n \]

But:

\[ \begin{align*}
\int dx_1 dx_2 z_2 &= e^{4k \log \frac{L}{2}} \quad y^n \\
\int dx_1 dx_2 z_2 &= e^{4k \log \frac{L}{2}} \quad y^n \\
\end{align*} \]

Similarly:

\[ \int dx_1 dx_2 z_2 \]

So:

\[ z_1 = e^{4k \log \frac{L}{2}} \quad z_2 + \int dx_1 z_1 + \int dx_2, dx_2 z_2 + \ldots \]

\[ = e^{4k \log \frac{L}{2}} \sum_{n=0}^{\infty} \frac{y^n}{n!} \]

\[ = \exp \left( 4k \log \left( \frac{L}{2} \right) \left[ 1 - y^2 \frac{L}{2} \right] \right) \]
Redefine $\gamma \rightarrow \sqrt{a} \gamma$

$e' = k' = k(1-\gamma^2\ell)$

$$\frac{dk}{d\ell} = -ky^2$$

But, there are 4 blips which affects the interactions between two kinks:

So, we need a factor of four:

$$\frac{dk}{d\ell} = -4ky^2$$

2nd R.G. flow equation