The question we would like to answer: Do superfluids exist in 2D?

We usually associate SF with 3D long-range order (LRO)

\[ \langle \Psi(r) \Psi(0) \rangle \sim \text{const. as } r \to \infty \]

o.p. for superfluid.

In 1966, Memin–Wagner: 2D systems w/ continuous symmetry \( \Rightarrow \) NO LRO.

Even before 1966, Bloch: no spot-

density for \( T > 0 \) in 2D may lattices


Mystery: Is superfluidity the same as LRO in all systems?

Two superfluids we study in lab

- \( ^4 \text{He} \) 3D \( T_c = 2.2 \text{ K} \), boson
- \( ^3 \text{He} \) 3D \( T_c = 1 \text{ mK} \), fermion

\( \rightarrow \) low \( T \): exotic cooper pair (boson).

believe SF associated w/ BEC

most properties: condensate wavefunction

\[ \Psi(r) = | \Psi | e^{i \theta(r)} \]

↑ fixed mag  ↓ phase can vary

\[ | \Psi | = \sqrt{n_s} = \sqrt{\rho_s / m} \]

\( \approx \) const.

- \( \theta \) is o.p. for 2D
  - gauge symmetry
  - \( \langle \theta(r) \theta(0) \rangle \)

- If we had a 2D SF, we would expect o.p.

  - One more important result

  - Show \( \rho_s \) looks like stiffness in lattice model

QM: probability current \( J \) (density)

\[ \nabla \times \vec{J} = \frac{\hbar}{2mi} [\Psi^* \nabla \Psi - \Psi \nabla^* \Psi] \]

\[ = \frac{\hbar}{m} \text{Im} [\Psi^* \nabla \Psi] = \frac{\hbar}{m} \text{Im} [(i \epsilon \nabla \theta) \times (i \epsilon \nabla \theta)] \]

\[ \vec{V}_s = \frac{\hbar}{m} \nabla \theta \]

\[ KE = \int d^2r \frac{1}{2} \rho_s | \nabla \theta |^2 \]

\[ = \int d^2r \frac{1}{2} \left( \frac{\rho_s \hbar^2}{m^2} \right) | \nabla \theta |^2 \]

\[ \approx \int d^2r \frac{1}{2} K_s | \nabla \theta |^2 \]

- Looks a lot like the energy for 2D XY model.
- If we had a 2D superfluid, we would expect that a good model would be

\[ 2D \text{ XY Model} \]

- simplest continuous symmetry is that of rotations in 2D plane
- order parameter that breaks that symmetry is

\[ \langle \vec{S} \rangle = S_0 \left( \cos \theta, \sin \theta \right) \]

- equivalently, a complex # of the type introduced in last section (fixed \( \vec{n} \), \( \theta = \theta(r) \)).
- imagine these spins on a lattice with spacing \( \vec{r} \)

\[ -\beta H = K \sum_{ij} \vec{S}_i \cdot \vec{S}_j \]

\[ = K \sum_{ij} \cos (\theta_i - \theta_j) \]

\[ \approx 1 - \frac{1}{2} (\theta_i - \theta_j)^2 \]

\[ \approx 1 - \frac{1}{2} a^2 (\nabla \theta)^2 \]

\[ \text{continuum limit} \]

and \( \sum_i \rightarrow \int d^2r \) where \( r \) is measured in units of \( a \)

\[ \text{so } \int d^2r (\star \cdot \star) = \frac{1}{a^2} \int dr d\theta r \]

\[ \text{Motiv: calculate energy of spin configurations} \]

\[ \text{This leads to} \]

\[ -\beta H = \beta E_0 - \frac{1}{2} \int d^2r \left( \frac{K}{2} (\nabla \theta)^2 \right) \]

\[ E_0 \text{ corresponds to all spins aligned.} \]

\[ E = \int d^2r \left( \frac{J}{2} (\nabla \theta)^2 \right) \]

as promised, energy of XY model

\[ \nabla \theta: \]

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\[ \text{spin wave} \quad \text{(\# vortex)} \]

What do experiments say?

Now slide show.
- Many experiments have explored thicker and thinner films in an attempt to answer our question.

- These culminated in the Torsional Oscillator measurements of Bishop + Reppy in 1978.

- These experiments were based on a very old idea due to Andronikashvili, which he used to measure the superfluid density in 1978.

- Makes use of the idea that liquid helium (below $T_c$) is comprised of two components, SF and normal.

- Implies...

- Change in resonant period $\Delta P \sim$ change in $\mathcal{W}$, ratio of fluid not sustained $\sim SF$.

- Modern version of T.O. with $Q \gg 10^6$ in $E$ able to resolve changes better than part in $10^9$.

- How do they make a SF film?
  - Extremely thin - a few atomic layers can resolve SF density to a part in $10^4$.

- What do they find?

- Describe

- Point out peak + rounding to features of dynamic Neq. ANNS.

Better punch line.

**EXPERIMENTAL ANSWER** regardless of whether there is a theory, is that there is SF in 2D.

- Maybe SF is not connected to LRO in all superfluid systems.
What is special about 2D Weirdness of 2D

Recall:
\[-\beta H = -\int d^2 r \frac{K}{2} (\nabla \Theta)^2\]

would like to calculate
\[G(r) = \langle \Theta(r) \Theta(0) \rangle\]
easiest w/ F.T.
\[\Theta(r) = \frac{1}{(2\pi)^2} \int d^2 q \ e^{i q \cdot r} \tilde{\Theta}_q\]

so
\[\frac{K}{2} \int d^2 r |\nabla \Theta|^2 = \frac{K}{2} \frac{1}{(2\pi)^2} \int d^2 q d^2 q' (i q) \tilde{\Theta}_q \tilde{\Theta}_{q'} \int d^2 r e^{i(q-q') \cdot r} \]

Hamiltonian in fourier space:
\[-\beta H = -\frac{1}{(2\pi)^2} \frac{K}{2} q^2 |\tilde{\Theta}_q|^2\]

uncoupled oscillators: each DOF has \(\frac{1}{2} k_B T\) equi-partition:
\[\frac{K}{2} q^2 \langle \tilde{\Theta}_q^2 \rangle = \frac{1}{2}\]
\[\langle \tilde{\Theta}_q^2 \rangle = \frac{1}{K q^2}\]

\[G(r) = \langle \Theta(r) \Theta(0) \rangle = \frac{1}{(2\pi)^4} \int d^2 q d^2 q' e^{i q \cdot r} \langle \tilde{\Theta}_q \tilde{\Theta}_{q'} \rangle\]

It turns out all physical correlation functions given in terms of \(\frac{1}{K q^2} \delta(q-q')\)

\[G(r) - G(0) = \frac{1}{(2\pi)^2} \int d^2 q \frac{1 - e^{i q \cdot r}}{K q^2}\]

would like \(r \to \infty\) limit, see that \(e^{i q \cdot r}\) is oscillatory and cancels

\[\approx -\frac{1}{2\pi K} \int \frac{2\pi q dq}{q^2} \frac{1}{K q^2}\]

leading behavior for \(r \to \infty\)

\[= -\frac{1}{2\pi K} \log \left(\frac{r}{a}\right) + \frac{c}{K} + \ldots\]

And spin-spin corr. has to be on dimensional grounds

\[\langle S(r) \cdot S(0) \rangle = \langle \cos(\Theta(r) - \Theta(0)) \rangle\]

\[= \text{Re} \langle e^{i [\Theta(r) - \Theta(0)]} \rangle\]

trick from Cauchy-Gaussian integration

\[\langle \psi(r) \psi^* (0) \rangle = |\psi|^2 \langle e^{i [\Theta(r) - \Theta(0)]} \rangle\]

and go straight to

\[e^{-\frac{i}{2} \langle \Theta(r)^2 \rangle} G(r) = e^{-\frac{i}{2} \langle \Theta(r)^2 \rangle} e^{-\frac{i}{2} \langle \Theta(0)^2 \rangle} e^{-\frac{i}{2} \langle \Theta(r) \Theta(0) \rangle} G(r) = e^{-\frac{i}{2} \langle \Theta(0)^2 \rangle} e^{-\frac{i}{2} \langle \Theta(r)^2 \rangle} G(r)\]
- we just showed M-W Thm.

\[
\text{low } T \quad \rightarrow \quad \text{high } T
\]

- O.P. corr. die
- algebraic
- \(\exp \left( \Gamma - \frac{1}{2\pi K} \right)\)

- unique type of phase transition
  \(Q \rightarrow \text{disorder}\)
  or \(P_s\).

- \(K = K(T)\) reduced by vortices
  as \(T \rightarrow T_c\).

**part II:** do a renormalization calculation to find out how.
Vortices

- Some intuition - topological defect in the field \( \phi \) characterized by some region of size \( a \) (core) where order is destroyed.

- Need to know form of \( G(r) \), for \( \theta \):

\[
\int \nabla \theta \cdot dl = 2\pi 
\Rightarrow \quad |\nabla \theta| \cdot 2\pi \cdot r = 2\pi 
\Rightarrow \quad \nabla \theta = \frac{1}{r} 
\]

- Energy of vortex

\[
E_{\text{vor}} = \int d^2r \frac{J}{2} (\nabla \theta)^2 
= 2\pi \int_a^L dr \cdot r \frac{J}{2} \left( \frac{1}{r^2} \right) 
= \pi J \ln \left( \frac{L}{a} \right) 
\]

- Energy of vortex is logarithmic in size of system (in 2D).

- At low \( T \), \( E \) dominates free energy, and so prob. of having an isolated vortex is small (and zero in the thermodynamic limit \( L \to \infty \)).

- This led K and T to a simple argument for a transition to a high-\( T \) state where an isolated vortex would be energetically more favorable.

\[
\text{Free energy change, sign at } T_c = \frac{\pi}{2} J 
\]

\( \text{low } T: \) no vortex  \( \text{high } T: \) vortex energetically favorable to have a vortex at \( T > T_c \)

\( \text{"Universal Jump" } \)

\[
\frac{J}{T_c} = \frac{2}{\pi} 
\]

or

\[
\frac{\rho_s(T_c)}{T_c} = \frac{2}{\pi} \cdot \frac{m^2}{\hbar^2 k_B} 
\]

- All systems very have K-T transition have this property.

Show slides again - colored.

Experimentally: tune \( \rho_s(T_c) \).

- Although can't read it off, \( \rho_s/T_c \) is a parameter of these fits.

- Actually has a square-root cusp ~ \( \sqrt{T_c - T} \), which is a prediction of nice RG flows.

- Repy + Bishop fit .... other results from \( \text{3rd sec.} \).
Vortex Pairs

- In reality, not stable: no vortices \( \to \) isolated vortices

\[
E_{\text{vor}} = 2E_c + E_1 \ln \left( \frac{R}{a} \right)
\]

- stable: finite energy

- \( L \) drops out.

- more and more vortex pairs of all sizes, more as Temp increases

- Insulator: dipoles change \( E \)

- 2D SF: vortex pairs change \( K \)

(or \( p_x \)).

- can think of vortices as mobile degrees of freedom that arrange themselves to minimize free energy if we impose a macroscopic \( \nabla \Theta_{\text{ext}} \) \( \to \) a velocity for SF

- In other words: vortices screen ext. field

\[
F(\nabla \Theta_{\text{ext}}) - F(0) = \frac{1}{2} V K^R (\nabla \Theta_{\text{ext}})^2
\]

- effect is that macroscopic \( K^R \neq \) microscopic \( K \)

- Nice to have a picture of the screening e.g.

fig 9.4.1 of Chaimin + Lubensky