Flocking / Nonlinear Hydrodynamics

Flocking: The collective, coherent motion of a large number of organisms

\[ \rightarrow \text{Universal Property: seen in length scales from } \mu\text{m (Dictostelium discoideum) to m (birds) to km (wildebeasts)} \]  

\[ \rightarrow \text{Show Starling Movie (if rods are available)} \]  

Two Types of Flocking:
1) Symmetry Broken by external field (compass)  
2) Symmetry Broken by interactions w/ neighbors  

We are more interested in case #2 (Mean-Field)  

Big Question: Given no external field exists, how does symmetry breaking occur?  

\[ \rightarrow \text{will use RG on a continuum theory of motion} \]  

We can think of flocking analogously to a ferromagnet

\[
\begin{align*}
\text{Magnet:} & \quad \text{Flock:} \\
\text{spin:} & \quad \text{velocity:} \\
M & \quad \langle \mathbf{v} \rangle \\
\text{external field:} & \quad \text{misalignment error} \\
T & \quad \text{will flip here} \\
\text{Both have only short-range interactions} & 
\end{align*}
\]
Problem: How can we get symmetry-breaking in a system with rotational invariance, only short-range interactions, and finite T? (M-W says that it's impossible)

→ We'll see

Microscopic Model (Vicsek et al., 1995)

1) N "boids" in volume L^d (ρ = N/L^d) w/ periodic B.C.

2) All boids move w/ constant speed V_0 & interact only with other boids within a radius R_0.

3) Heading = θ_i & θ_i^{t+1} = <θ_j^{t,neighbors} + η_i^t

4) <η_i(t)η_j(t')> = Δδ_{ij}δ(t-t') (Δ ↔ T)

Note: XY model if V_0=0, Mean Field Theory if V_0→∞

→ Show Microscopic Movies

for d=2:

\[ |\vec{v}| \sim (\Delta - \Delta_c)^{0.45} \]

\[ |\vec{v}| \sim (\rho - \rho_c)^{0.35} \]

Long-range order in 2D!

We'll analyze by going to a continuum theory
Hydrodynamic Model

Basic Idea: Write down a PDE which contains all relevant terms allowed by the system's physics

→ Too complicated to proceed by decimating the microscopic model (a la 1-D Ising)

Specifically:
1) Write down all relevant terms
2) Eliminate terms disallowed by symmetries/conservation laws

Example: Navier-Stokes Equations

\[ 10^{23} \text{ molecules} \rightarrow \vec{u}, \rho, T, D \]

Conservation laws
- Mass
- Momentum (second)
- Energy

Symmetries
- Translation
- Rotation
- Galilean (first)

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \rightarrow \text{Typical continuity equation}
\]

\[
\frac{\partial \vec{v}}{\partial t} + \nabla \cdot (\vec{v} \otimes \vec{v}) = \frac{1}{\rho} \left( -\nabla \mathcal{P} + \frac{\mu}{\rho} \nabla^{2} \vec{v} + \frac{\tau}{\rho} \right)
\]

\[
\mathcal{P}(\rho) = \sum_{n=1}^{\infty} (\rho - \rho_0)^n \quad < f_n(\vec{r} \pm t) f_j(\vec{r} \pm t') > = \Delta s(t) \delta(\vec{r} - \vec{r}') \delta(t - t')
\]
So what does this tell us? (Eqs are hard)

When do we have an ordered state?

→ Steady-state & Uniform Flow case \((\frac{\partial \vec{v}}{\partial t} = 0, \frac{\partial \vec{v}}{\partial \vec{x}} = 0)\)

\[ 0 = \alpha \vec{v} - \beta |\vec{v}|^2 \vec{v} \]

\[ |\vec{v}| = \sqrt{\frac{\alpha}{\beta}} \equiv V_0 \]

→ Ordered flow exists only when \(\alpha, \beta\) have the same sign (but is it stable?)

Previous week → All long range order gets quenched by fluctuations

Quasi-Linear Theory

We need to make some simplifications to make eqns tractable

\(\hat{\vec{v}} = V_0 \hat{\vec{x}} \parallel + \hat{\vec{v}} = (V_0 + \delta V_\parallel) \hat{\vec{x}} \parallel + \hat{\vec{v}}\)

→ Assume fluctuations in \(\delta V_\parallel\) are linear, then plug back into full eqns to see what happens with \(\vec{v}\)

\[ \frac{\partial \hat{\vec{v}}}{\partial t} + \chi_1(\ ) + \chi_2(\ ) = - \hat{\vec{v}} \hat{\vec{P}} + \hat{\vec{D}}(\ ) + \hat{\vec{D}}(\ ) + \hat{\vec{D}}(\ ) + \hat{\vec{D}}(\ ) \]

\[ \text{nonlinear terms!} \]

(Similar Mass Conservation
(Assumes \(\rho = \rho_0 + \delta \rho\))
First Step: Linearize in $\vec{v}$ & look at $\langle \vec{v}(\vec{r},t) \rangle^2$

If this diverges, fluctuations dominate

$$<\vec{v}(\vec{r},t)^2> = \int \frac{d^d q}{(2\pi)^d} C_{ii} \sim \int \frac{d^d q}{q^2} \sim \int \frac{d^d q}{q^2}$$

$\Rightarrow$ Diverges in the UV $d > 2$ ($q \to \infty$)
$\Rightarrow$ Infared $d \leq 2$ ($q \to 0$)

Note: UV divergence is troublesome, since we don't expect the theory to hold at small wavelengths
2) Infared divergence $\Rightarrow$ Fluctuations win on long wavelengths

$\Rightarrow$ No flocking in 2D! (MW strikes again)

$\therefore$ Flocking must be a non-linear effect.

Scaling Analysis:

$X_{\perp} = b X_{\perp}$
$X_{\parallel} = b^z X_{\parallel}$ $\rightarrow$ Anisotropy exponent
$t' = b^z t$ $\rightarrow$ Time exponent
$\vec{V}_1 = b^z \vec{V}_1$ $\rightarrow$ Roughness exponent (determines if fluctuations take over)

$\Rightarrow$ In linear theory, only diffusion constants $\xi$ noise are important
$\Rightarrow$ Keep them constant under rescaling ($\xi$-expansion)

$D_{\perp} = b^{2-z} D_{\perp} \Rightarrow z = 2$
$D_{\parallel} = b^{z-2\xi} D_{\parallel} \Rightarrow \xi = 1$
Scaling (cont.)

To keep the form of the f-f correlations
\[ f' = b^{-1 - d/2} f \]
Wait noise to remain constant w.r.t. linear terms:
\[ \frac{\partial^2 \psi}{\partial z^2} = b^{-2} \frac{\partial \psi}{\partial t} \Rightarrow \chi - z = -1 - d/2 \]
\[ \Rightarrow \chi = 1 - \frac{d}{2} \]  
Again, no flocking in d = Z since Z fluctuations don’t die

Nonlinear Terms:
\[ \lambda_1, \lambda_2 \rightarrow b^{Z - d/2} \Rightarrow \lambda \]  
\[ \sigma \rightarrow b^{n + (1-n)d/2} \]  
(4 is the upper critical dimension)

But still, what happens for 2d < 4? Should have flocking in d = 2

\[ \Rightarrow \text{Dynamic RG on quasi-linear equations (allow all parameters to flow)} \]

\[ \text{Messy calculation, will outline} \]

1) Transform Q-L eqns into factor (squared)

2) LH the E.O.M. for the inner cylinder (hard!!) & apply them on the outer shell

3) rescale, keeping constant

\[ \Rightarrow \delta t, \delta, \xi \text{ s.t. the nonlinear terms } (\lambda_1, \lambda_2, \sigma, ...) \]
are fixed

\[ \Rightarrow \text{This is an expansion about the nonlinear fixed pt.} \]

\[ \Rightarrow \frac{d\epsilon}{d\xi}, \frac{d\sigma}{d\xi}, \ldots \]
1) Nonlinear terms scale identically to the "harmonic" analysis.

2) Due to calculational difficulties, only bounds can be put on exponents for $2 < d < 4$ ($x_2$ causes problems):

\[ \frac{6}{5} < z < 2, \quad \frac{3}{5} < \xi < 1, \quad \chi < \min\left(\frac{1}{3}, 1 - \frac{d}{2}\right) \]

3) Since $x_2 = 0$ for $d = 2$, exponents are known:

\[ z = \frac{6}{5}, \quad \xi = \frac{3}{5}, \quad \chi = -\frac{1}{5} \]

\[ \Rightarrow \text{For } 2 < d < 4, \quad \vec{V}_t' \sim b^\xi \vec{V}_t \]

since $\chi < 0 \Rightarrow$ Fluctuations die

\[ \Rightarrow \text{Long-Range Order} \]