\[ Z = \text{Tr} \ e^{\beta J \sum S_i S_j} \]
\[ = \text{Tr} \ \prod e^{\beta J S_i S_j} \]
\[ = \text{Tr} \ \prod \left[ \sum_{n\text{ even}} \frac{1}{n!} \left( \beta J S_i S_j \right)^n + \sum_{n\text{ odd}} \frac{1}{n!} \left( \beta J \right)^n S_i S_j \right] \]
\[ = \text{Tr} \ \prod \left[ \cosh \beta J + (\sinh \beta J) S_i S_j \right] \]
\[ \propto \text{Tr} \ \prod \left[ I + (\tanh \beta J) S_i S_j \right] \]

This has terms of the form

\((\tanh \beta J)^n S_i S_j \ldots S_{i_n} S_{j_n}\)

Since \(\text{Tr} \ S_i^n = 0\) if \(n\) is odd, only sets of \(S_i\)'s that form connected, closed paths survive. Hence some number of

\[ Z = \prod \left( \tanh \beta J \right) \text{length of paths} \]
To find a correlation function, we insert

\( \langle s_a s_b \rangle = \text{Tr} s_a s_b e^{\beta J_E s_a s_b} \)

Now the set of curves that survives the sum includes an open curve connecting \( s_a \) to \( s_b \).

This is like a domain wall on the dual lattice.

SLE is a method for describing such a domain wall.

What we need is a probability measure on curves in 2d that connect one point on the boundary of a set to another.

Physicists like the path integral measure. Mathematicians like Brownian motion. How can we relate these curves to Brownian motion?

Enter Karl Löwner.
Lowner invented a method for describing the growth of a \(2d\) curve from the boundary of the upper half plane. The idea was to find a conformal map of the form

\[
\begin{align*}
  b &\rightarrow b_+ \\
  a &\rightarrow a_+
\end{align*}
\]

This is easy for a straight line:

\[
\begin{align*}
  a &\rightarrow a + 2.1 \\
  a &\rightarrow a
\end{align*}
\]

Now imagine a growing curve with time \(t\) and a map \(g_t\) that maps it out.

\[
\begin{align*}
  g_t = a + \left(\frac{(2-a_2)^2 + 4t}{2}\right)^{1/2}
\end{align*}
\]

Now, we want to find a measure on \(2d\) Brownian curves. What properties should the measure have?

This measure is mathematically undefined and rigorous.
The map \( g dt \) that maps out the extra bit should be the straight line map:

\[
g_{++dt} = a_t + \sqrt{(g_t - a_t)^2 + 4ldt)}^{1/2}
\]

\[
\frac{d}{dt} g_t = \frac{2}{g_t - a_t}
\]

Now we have a conformal map that associates a 1D function \( a_t \) to a growing 2D curve. What if we want a random curve?

Now our measure on 2D curves will be a measure on 1D functions \( a_t \). What properties should this measure have?

1. Markov property:
   \[
   \mu(\gamma_2 | \gamma_1 , D) = \mu(\gamma_2 , D | \gamma_1)
   \]

2. Conformally invariant (critical curves)

3. Reflection invariance

A measure with these properties is the measure on 1D Brownian curves, \( B_t \). This measure is mathematically well defined and rigorous.
SDE

\[ \hat{g}_t = g_t - \alpha_t \]
\[ d\hat{g}_t = \frac{2d\hat{g}_t}{\hat{g}_t} - d\alpha_t \]
\[ = \frac{2d\hat{g}_t}{\hat{g}_t} = \sqrt{\kappa} \, dB_t \]

What is the significance of \( \kappa \)?

Aside on SDE:

\[ dx_t = ud\bar{t} + \nu dB_t \]
\[ \frac{dx}{dt} = u + \eta \], with \[ \langle \eta(t)\eta(t') \rangle = \nu^2 \delta(t-t') \]
\[ \langle dB_t \rangle = 0 \], \[ \langle dB_t^2 \rangle = dt \]

\[ dx_t = \frac{2dt}{x_t} + \sqrt{\kappa} \, dB_t \]

Say \( \kappa \) is small:

\[ x_t \, dx_t = 2dt \Rightarrow x_t^2 \sim 4t \]

Say \( \kappa \) is large:

\[ dx_t = \sqrt{\kappa} \, dB_t \Rightarrow x_t^2 \sim \kappa t \]
So for $K < 4$, the SLE curve is driven away from the origin, meaning that it does not intersect itself.

For $K > 4$, noise dominates and the trace will intersect itself (the real line) an infinite number of times. When this happens, the enclosed region is "swallowed" and mapped to a single point.

There is a duality between these two phases:

\[
\begin{align*}
\mathcal{L} & \quad \rightarrow \quad \mathcal{L} \\
\end{align*}
\]

Relating $K \leftrightarrow \frac{16}{K}$

As we will see, $K$ and $\frac{16}{K}$ correspond to the same CFT.
What can we calculate with SLE?

Geometrical events - e.g. probability of SLE hitting some set.

Examples:
- Left passage
- Fractal dimension
- Crossing probability

Associating SLE with physics models.

Restriction, locality

What does SLE do?

Outline - questions

Blackboard

[Diagram of a sketch with labeled points A, B, C, D, and the phrase 'Crossing probability']
What does the SLE do?

- Defines a measure on various classes of random curves in 2D (the measure on $\mathbb{R}^2$).

- Describes growth of curves in terms of an SDE that can be solved using rigorous methods.

- Exploits the conformal invariance and Markov properties of the measure to express the probabilities of various events as SDE's which can then be turned into PDE's that can be solved.