RG Derivation - Alex Moore, Maicol Daza

Start by writing a general vector field:

\[ \dot{x} = \sum_i a_i x^i \]

(1)

Now rescale time by \( b \), and rescale \( x \):

\[ t' = bt \]
\[ x' = b^{\lambda_x} x \]

(2)

(3)

Coarse graining time corresponds to setting \( b < 1 \). We now rescale equation (1):

\[ \dot{x} = \sum_i a_i (b^{-\lambda_x} x')^i \]

(4)

and write \( \dot{x} \) in terms of the rescaled dynamics, \( \frac{dx}{dt'} \):

\[ \dot{x} = \frac{dx}{dt'} \frac{dt'}{dt} = \frac{dx}{dt'} b = \frac{dx'}{dt'} b^{1-\lambda_x} \]

(5)

Finally putting equations (2) and (3) together, we get:

\[ \frac{dx'}{dt'} b^{1-\lambda_x} = \sum_i a_i (b^{-\lambda_x} x')^i \]

(6)

\[ \frac{dx'}{dt'} = \sum_i a_i b^{(1-i)\lambda_x-1} (x')^i \]

(7)

Expanding this out, we get:

\[ \frac{dx'}{dt'} = a_0 b^{\lambda_x-1} + a_1 b^{-1} x + a_2 b^{-(\lambda_x+1)} x^2 + a_3 b^{-(2\lambda_x+1)} x^3 + a_4 b^{-(3\lambda_x+1)} x^4 + ... \]

(8)

In general, if we want to fix the \( i \)th term, we will set \( \lambda_x = \frac{1}{i} \). This will give relevant variables for all \( x^k \) terms with \( k < i \), and all irrelevant variables for terms with \( k > i \).

**Fix \( x^0 \) term:**

To fix the \( x^0 \) term we need \( \lambda_x = 1 \). Plugging this in gives:

\[ \frac{dx'}{dt'} = a_0 + a_1 b^{-1} x + a_2 b^{-2} x^2 + a_3 b^{-3} x^3 + a_4 b^{-4} x^4 + ... \]

(9)

Since \( b < 1 \), this makes all the terms grow under coarse-graining. Thus all the terms are relevant variables.

**Fix \( x^1 \) term:**

To fix the \( x^1 \) term we need \( b = 1 \). This corresponds to no rescaling of time and returns back the original equations:

\[ \frac{dx'}{dt'} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + ... \]

(10)

**Fix \( x^2 \) term:**

To fix the \( x^2 \) term we need \( \lambda_x = -1 \). Plugging this in gives:

\[ \frac{dx'}{dt'} = a_0 b^{-2} + a_1 b^{-1} x + a_2 b^2 x^2 + a_3 b x^3 + a_4 b^2 x^4 + ... \]

(11)

The \( x^0 \) and \( x^1 \) both grow under coarse-graining making them relevant perturbations. The terms \( x^3, x^4 \) and higher all shrink making them irrelevant.
**Fix $x^3$ term:**

To fix the $x^3$ term we need $\lambda_x = -1/2$. Plugging this in gives:

$$\frac{dx'}{dt'} = a_0 b^{-3/2} + a_1 b^{-1} x + a_2 b^{-1/2} x^2 + a_3 x^3 + a_4 b^{1/2} x^4 + ... \quad (12)$$

The $x^0$ and $x^1$ both grow under coarse-graining making them relevant perturbations. The terms $x^3$, $x^4$ and higher all shrink making them irrelevant.