Sphere in $3N$-space ("$3N-1$ sphere")

Allowed States have Kinetic Energy $E$,

$$\sum \frac{p_i^2}{2m} = E$$

lie on surface of sphere in $3N$ dimensions.

Volume of Sphere of Radius $R$:

$$R = \sqrt{2mE}$$

Volume of Thin Shell $[E-\Delta, E] = \Omega(E) \cdot \Delta$

$$= \frac{\pi^{3N/2}}{(3N/2)!} \left(2m\right)^{3N/2} \left(E^{3N/2} - (E-\Delta)^{3N/2}\right)$$

$$\sim \frac{3N}{2} \Delta E^{3N/2-1} = \left(\frac{3N}{2} \Delta \frac{E}{E}\right) E^{3N/2}$$

Only $10^{23} \rightarrow$ will be irrelevant.

$$\rho(p) = \frac{1}{\text{Thin Shell Volume}} = \frac{1}{\Omega(E(p))}$$
What is \( P(P_x) \), the probability that atom 1 has \( x \)-momentum \( P_x \) ?

\[
\rho(p^* \Delta p) = \frac{\text{Area of "circle" disk}}{\text{Area of Sphere}}
\]

\[
\text{Disk is } 3N-2\text{-sphere} \quad \text{radius } R' = \sqrt{2mE - P_x^2}
\]

\[
\text{Area} = \frac{\pi^{\frac{3N-1}{2}}}{(3(N-1))!} \left( 2mE - P_x^2 \right)^{\frac{3N-1}{2}} \left[ \frac{3(N-1) \Delta \xi}{2E} \right] \cos \theta
\]

\[
= \frac{\pi^{\frac{3N}{2}}}{(3N/2)!} \left( 2mE - P_x^2 \right)^{\frac{3N}{2}} \left\{ \frac{\pi^{1/2}}{\xi \sqrt{N/2}}, \cos^2 \theta \right\}
\]

\[
\rho(P_x') = \left( \frac{\pi^{1/2} N^{3/2}}{2m} \right) \left( \frac{2mE - P_x^2}{2mE} \right)^{\frac{3N}{2}}
\]

\[
= \left( \frac{1}{\sqrt{2m}} \right) \left( 1 - \frac{P_x^2}{2mE} \right)^{\frac{3N}{2}}
\]

\[
\rho(P_x') = \left( \frac{\sqrt{2m}}{2m} \right) \left( \frac{3N}{2} \right) e^{-\frac{P_x^2}{2m(3N/2)}}
\]

\[
\lim_{B \to \infty} \left( 1 - \frac{a}{B} \right)^B = e^{-a}
\]

\[
N \approx 10^{23}, \quad \frac{a}{3 \sqrt{2}} = \frac{P_x^2}{2mE}
\]

\[
a = \frac{P_x^2}{2m} \left( \frac{3N}{2} \right)
\]
Define \( T = \frac{1}{k_B} \left( \frac{2E}{3N} \right) \) for ideal gas.

\[ p(p_x) = \frac{1}{\sqrt{2\pi mkT}} e^{-p_x^2/2mkT} \]

\[ \sigma^2 = mkT = \langle p_x^2 \rangle \]

\[ \frac{1}{\sqrt{2\pi}} e^{-p_x^2/2\sigma^2} \]

1. Kinetic energy in each component of the velocity = \( p_x^2/2m = k_B T/2 \). Equipartition Theorem.

2. Most of the surface area of a large-dimensional sphere is very close to the equator!

3. Probability of this "subsystem" having given state \( \sim e^{-\frac{p_x^2}{2mkT}} \sim e^{-\frac{\text{Kinetic Energy}}{kT}} \)

First example of a Boltzmann distribution.

4. Temperature is the cost of stealing energy from the rest of the world.

   - Like pressure is energy cost for stealing volume
   - "Rest of the world" often called the heat bath

5. What currency is being paid?

Call \( S(E-K) \) = Volume of "circle" of radius \( \sqrt{2m(E-K)} \):

\[ \frac{1}{T} = \frac{d \log S}{dK} = \frac{dS}{dE} \]

\[ S = k_B \log S = \frac{3N}{2} k_B \log (2mE) + \text{const} = \text{Entropy} \]