LANDAU THEORY

Free Energy, Superfluids, Superconductors, Magnets, Martensites, Liquid Crystals, Dynamics, Surface Growth (HPZ), Crack Growth (Hodgeon, Eastgate), etc.

1. Pick an order parameter field.
   - Diffusion: scalar \( p \).
   - Waves on a string: height \( y \); Dynamics.
   - Magnets: vector \( M \).
   - Superfluid: complex number \( \Psi = \Phi e^{i\xi} \).

   • Allow change of magnitude as well as direction: Landau order parameter.
   • Good near phase transitions.
   • Useful for studying internal structure of defects (uncontrolled usually).

2. Write the most general possible free energy \( F(\Psi, \dot{\Psi}, M, \dot{M}, \frac{\partial M}{\partial x}, \frac{\partial M}{\partial t}, \frac{\partial \Psi}{\partial x}, \frac{\partial \Psi}{\partial t}, \frac{\partial^2 \Psi}{\partial x^2}, \frac{\partial^2 \Psi}{\partial t^2}, \frac{\partial^2 \Psi}{\partial x \partial t}, \ldots) \) or equation of motion

   \[ F(\Psi, \dot{\Psi}, M, \dot{M}, \frac{\partial M}{\partial x}, \frac{\partial M}{\partial t}, \frac{\partial \Psi}{\partial x}, \frac{\partial \Psi}{\partial t}, \frac{\partial^2 \Psi}{\partial x^2}, \frac{\partial^2 \Psi}{\partial t^2}, \frac{\partial^2 \Psi}{\partial x \partial t}, \ldots) = 0 \]

   • Arbitrary, nonlinear function
   • Assumed local: long-range forces should have their fields as part of the order parameter.
3. Specialize to long length & time scales

\[ \frac{\partial W}{\partial x^N} \sim \frac{1}{D^N} \quad \text{as } D \to \infty \]

- Throw out high derivatives,

4. Impose the symmetries and conservation laws

Magnet: free energy

\[ \mathcal{F}(\vec{M}, \vec{x}, \frac{\partial M}{\partial x}, \frac{\partial^2 M}{\partial x^2}, \frac{\partial M}{\partial y}, \frac{\partial M}{\partial z}, \ldots) \]

- Translational symmetry: \( \vec{x} \to \vec{x} + \vec{\Delta} \) unchanged

- Rotational symmetry: Scalars from \( \vec{\nabla} \vec{M} \)

\[ \nabla \cdot \vec{M} \quad \left( \partial_x M_y \right) \left( \partial_x M_y \right) \quad \vec{M}^2, \quad \vec{M}^4 \]

\[ \mathcal{F}[\vec{M}] = A + B \vec{M}^2 + C \vec{M}^4 + D \nabla \cdot \vec{M} + E \partial_x M_y \partial_x M_y \]

\[ + F \left( \partial_x M_y \partial_y M_y \right) + G \left( \partial_x M_y \partial_y M_y \right) + M \partial_x \partial_y M_y \]

- Total Divergences

\[ \int \mathcal{F}[\vec{M}] \, d^3x = \ldots + \int \text{Divergence} \]

Integrate by Parts

\[ \int \partial_x M_y \partial_x M_y \, dx = \int M \partial_x \partial_y M_y \, dx \]

\[ = \left[ \text{Boundary terms} \right] + \int \partial_x M_y \partial_y M_y \, dx \]

\[ \mathcal{F}[\vec{M}] = A + B \vec{M}^2 + C \vec{M}^4 + E \partial_x M_y \partial_x M_y + F \left( \partial_x M_y \right)^2 \]
Wave Equation - Stretched String

Most general equation of motion, $y(x,t)$, up to second derivatives:

$$F(y, x, t; \frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}, \frac{\partial^2 y}{\partial x^2}, \frac{\partial^2 y}{\partial t^2}, \frac{\partial^2 y}{\partial x \partial t}) = 0$$

Translation Invariance $\rightarrow \times$

Time independence $\rightarrow$ not $t$

Shift string upward $\rightarrow$ no $y$

Small displacements $\rightarrow$ linear in $y$

$$A \frac{\partial y}{\partial x} + B \frac{\partial y}{\partial t} + C \frac{\partial^2 y}{\partial x^2} + D \frac{\partial^2 y}{\partial t^2} + E \frac{\partial^2 y}{\partial x \partial t} = 0$$

Spatial inversion $\rightarrow A = E = 0$

Time reversal invariance (no friction) $\Rightarrow B = 0$

$$C \frac{\partial^2 y}{\partial x^2} + D \frac{\partial^2 y}{\partial t^2} = 0$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{C}{D} \frac{\partial^2 y}{\partial x^2} \quad \text{Wave equation} \quad v = \sqrt{\frac{C}{D}}$$
Diffusion Equation

- \( \rho \) is conserved.

\[
\frac{\partial \rho}{\partial t} = - \nabla \cdot \mathbf{J}
\]

- Most general:
  - Long Length & Time
  - High Gradients
  - Space & Time Translations

\[
\mathbf{J}(\rho, \mathbf{x}, t, \nabla \rho, \partial \rho / \partial t, \ldots)
\]

\[
= -D(\rho) \nabla \rho \quad \text{(Must have one gradient)}
\]

\[
\frac{\partial \rho}{\partial t} = - \nabla \cdot (D(\rho) \nabla \rho)
\]

\[
= D \nabla^2 \rho \quad \text{if } \rho \approx \text{constant}
\]
1. **Why are power series special?**

   Why not $A \sqrt{m^2} + B \Theta(|m^1 - 6|) + \ldots$?

   - Local coarse-grained free energies always analytic

   $Z = \text{Tr}(e^{-\beta H}) = \sum e^{-\beta H}$
   
   Eigenvalues
   
   Small region $\Rightarrow$ discrete eigenvalues
   
   Each term analytic
   
   $\Rightarrow$ Any finite-size system has analytic free energy

   Usually same for

   $e^{-\beta F[M]} = \text{Tr} e^{-\beta H}$
   
   $\{ \text{states with } M(x) \}$
   
   Local

   - Not always true for equations of motion!
   
   Non-equilibrium

   Plasticity? (Our current research).
- Noise, Dirt

Statistical Mechanics. Fluctuations often are crucial to the behavior, must be included in the free energy &/or laws of motion.

→ Second Order Phase Transitions
  Thermal Fluctuations Change the Behavior

  Critical Phenomena, Renormalization Group (still starts with Landau theory).

→ Fluctuation-dominated Phases
  Spin glasses: dirt important for all $T < T_c$
  KPZ: fluctuations cause roughness on all scales.