Liouville's Theorem

Let's consider the evolution law for a general probability density in phase space

\[ P(q_1, \ldots, q_{2N}, p_1, \ldots, p_{2N}) \]

\[ \dot{q}_m = (q_{3m-2}, q_{3m-1}, q_{3m}) \]

\[ \dot{p}_m = (p_{3m-2}, p_{3m-1}, p_{3m}) \]

We know that probability density is locally conserved; the change in a region equals the flow through the boundary. Using Gauss' law in 6N dimensions,

\[ \frac{\partial P}{\partial t} + \text{div}(PV) = 0 \]

Flow out = density \cdot velocity

\[ \frac{\partial P}{\partial t} + \sum_{\alpha=1}^{3N} \frac{\partial P}{\partial q_{\alpha}} \dot{q}_{\alpha} + P \frac{\partial {\dot{q}}_{\alpha}}{\partial q_{\alpha}} + \frac{\partial P}{\partial p_{\alpha}} \dot{p}_{\alpha} + P \frac{\partial {\dot{p}}_{\alpha}}{\partial p_{\alpha}} = 0 \]

Product Rule

So far, everything is true for any dynamics. We have a special case: no friction, no forcing: Hamiltonian dynamics

\[ \dot{q}_{\alpha} = \frac{\partial H}{\partial p_{\alpha}}, \quad \dot{p}_{\alpha} = -\frac{\partial H}{\partial q_{\alpha}} \]

\[ \frac{\partial^2 H}{\partial q_{\alpha} \partial p_{\alpha}} = \frac{\partial H}{\partial q_{\alpha}} \frac{\partial H}{\partial p_{\alpha}}, \quad \frac{\partial H}{\partial p_{\alpha}} \frac{\partial H}{\partial q_{\alpha}} = \frac{\partial H}{\partial p_{\alpha}} \left( -\dot{p}_{\alpha} \right) = -\frac{\partial^2 H}{\partial q_{\alpha} \partial p_{\alpha}} \]

\[ \frac{\partial P}{\partial t} + \sum_{\alpha=1}^{3N} \frac{\partial P}{\partial q_{\alpha}} \dot{q}_{\alpha} + \frac{\partial P}{\partial p_{\alpha}} \dot{p}_{\alpha} = \frac{\partial P}{\partial t} = 0 \]

Liouville's Theorem
What is $\frac{dp}{dt}$?

It's the time evolution of $p$ seen by a particle moving with the flow.

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + \nabla p \cdot \vec{V}$$

- Change in $p$ due to motion
- Volume $\Delta p \cdot (\vec{V} \Delta t)$
- Fixed position $\frac{dp}{dt} \Delta t$

What does it mean?

- Volume in phase space is conserved.

$P = \frac{1}{\Delta x \Delta p}$ inside box

Volume same, perhaps stretched, twisted, folded

- No "attractors" in Hamiltonian systems on energy surface
- Uniform density in microcanonical ensemble stays uniform. [Preferred measure on phase space: symplectic form $\omega = dp \wedge dq$]

- With ergodic hypothesis, the only stationary $p$ are those constant on each energy surface. (Trajectory covers energy surface, $p$ constant on trajectory $\Rightarrow P(E)$ function of $E$ only.)