The Ideal Gas

$N$ "non-interacting" monatomic atoms confined to box, volume $V$

For today, let's assume energy is conserved when atoms bounce off walls.

- Each atom retains initial velocity? Bad. Assume weak interactions between atoms, dilute gas, wait for long time until velocities are scrambled.

How do we extract simple behavior out of complex, microscopic trajectories?

Assume trajectory visits all allowed states equally often.

$\Rightarrow$ Average properties over all states with fixed total energy.

$$\langle \text{Time Average} \rangle = \langle \text{Phase Space Average} \rangle$$
Boltzmann's Ergodic Theorem:

Over a sufficiently long time, the trajectory of a closed system in phase space (positions and momenta) passes arbitrarily close to every point on the surface of constant energy (and other conserved quantities).

Not True for Planets!

HW 1.3: KAM Theorem

What is the probability density \( p(\mathbf{X}) \) that the atoms will be in a particular configuration \( \mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \mathbb{R}^{3N} = \mathbb{X} \)?

\[
\int p(\mathbf{X}) \, d^3\mathbf{x}_1 \, d^3\mathbf{x}_2 \cdots d^3\mathbf{x}_N = 1
\]

\[\rightarrow p(\mathbf{X}) = \frac{1}{\mathcal{Z}} = \frac{1}{V^N}
\]

Distinguishable particles


\( \mathcal{Z} = V^N / N! = \text{Total configuration space volume of distinct arrangements} \)

\( p(\mathbf{X}) = \frac{N!}{V^N} \)
$2^{-N} \binom{N}{M} = 2^{-N} \frac{N!}{M!(N-M)!} = \text{Probability } M \text{ atoms on left half}$

Why?

- Total \# of subsets of set of $N$ distinguishable atoms
  $= 2^N$ (each atom either in or out)
  each has probability $2^{-N}$.

- Total \# of ordered subsets of size $M$
  $\binom{N}{M} = \frac{N!}{(N-M)!} = \frac{N!}{(N-M)!(M)!}$
  # of ways of ordering atoms on right

- Total \# of subsets of size $M$, indistinguishable
  $\binom{N}{M} = \binom{N}{M}$

- Probability $M$ atoms on left $= 2^{-N} \binom{N}{M}$
Exercise: What is the probability that all $N$ atoms at time $t$ will be on the left half of the box?

$$2^{-N}$$

$$2 = 10^{2.3} \cdot 0.85 \cdot 10^{2.3} = 10^{3.85} \cdot 0.85 = 10^{3.85} \cdot 0.85$$

Exercise: What is the probability that $(N/2+m)$ atoms are on the left side?

$$P_m = 2^{-N} \binom{N}{N/2+m} = 2^{-N} \frac{N!}{(N/2+m)!(N/2-m)!}$$

Let's simplify using Stirling's formula:

$$\log n! \approx n \log n - n + \frac{1}{2} \log(2\pi n) + \cdots$$

Fixes prefactor, ignore for now

$$\log P_m = -N \log 2 + (N \log N - N)$$

$$- \left( \frac{N}{2} \log \left( \frac{N}{2} \right) - \frac{N}{2} \right) - \left( \frac{N}{2} - m \right) \log \left( \frac{N}{2} - m \right) - \left( \frac{N}{2} + m \right) \log \left( \frac{N}{2} + m \right)$$

$$\log N \approx \log 2 + \log \left( 1 + \frac{2m}{N} \right)$$

$$\log (1+e) \approx e - \frac{e}{2}$$

$$= -N \log 2 + N \log N - \left( N \log N - N \log 2 + \frac{N}{2} \left[ \log \left( 1 + \frac{2m}{N} \right) + \log \left( 1 - \frac{2m}{N} \right) \right] \right)$$

$$+ m \left[ \log \left( 1 + \frac{2m}{N} \right) - \log \left( 1 - \frac{2m}{N} \right) \right]$$

$$= \frac{N}{2} \left( 2 \left( \frac{2m}{N} \right)^2 \right) - m \left( 2 \frac{2m}{N} \right) = -\frac{2m^2}{N}$$

$$P_m \approx e^{-\frac{2m^2}{N}}$$

Relative Fluctuations $\frac{\sigma}{N^{1/2}} \approx Normalization$ $\Rightarrow \sigma = \sqrt{N/2}$

$$\frac{1}{N^{1/2}} \approx 10^{-11} = 0.000000001\%$$