This week the material is more formal and theoretical. The pre-class exercises are more challenging than normal – you may wish to use Mathematica for some of the integrals, although all can be done by hand using contour integration.

Pre-class Preparation


Wednesday
Read: Chapter 10, Sec. 10.3 (Equal-time correlations in the ideal gas) and 10.4 (Onsager’s regression hypothesis and time correlations)
Pre-class question: 10.12: *Liquid free energy.* (Submit electronically by 9:30 Tuesday evening.)

Friday
Read: Chapter 10, Sec. 10.5 (Susceptibility and linear response) and 10.6 (Dissipation and the imaginary part)
Pre-class question: 10.14: *Liquid susceptibility and dissipation.* (Submit electronically by 9:30 Thursday evening.)

Monday
Read: Chapter 10, Sec. 10.7 (Static susceptibility), 10.8 (The fluctuation-dissipation theorem), and Sec. 10.9 (Causality and Kramers-Krönig)
Pre-class question: 10.16: *Static susceptibility and the fluctuation–dissipation theorem for liquids,* and 10.18: *Kramers Krönig for liquids.*
(Submit electronically by 9:30 Sunday evening.)

Exercises
Those in 4488 may choose two of the four exercises.

10.5: *Telegraph noise in nanojunctions.*
10.20: *Subway bench correlations.*
10.19: *Susceptibilities and correlations.*
10.8: *Magnet dynamics.*
In-class exercises

Day 1

10.10 Human correlations. 
Consider a person-to-person correlation function for the horizontal center-of-mass positions \( h_\alpha = (x_\alpha, y_\alpha) \) of people labeled by \( \alpha \), in various public places. In particular, let \( C_{\text{crowd}}(r) = \langle \rho(h) \rho(h + r) \rangle \), where \( \rho(h) = \sum_\alpha \delta(h - h_\alpha) \), and the ensemble average \( \langle \ldots \rangle \) is over a large number of similar events.

(a) Sketch a rough, two-dimensional contour plot for \( C_{\text{crowd}}(r) \), for a subway car of length \( L = 10m \) with two benches along the edges \( x = 0 \) and \( x = 3m \). Assume the people are all seated, with random distances \( \delta y = 0.5 \pm 0.2m \) between neighbors.

(b) Sketch a rough contour plot for \( C_{\text{crowd}}(r, t) \) in the same subway car, where \( t = 20 \) minutes is long enough that most people will have gotten off at their stops.

(c) Sketch a rough contour plot for the Fourier transform \( \tilde{C}_{\text{crowd}}(k) \) for the correlation function in part (a).

Day 1

7.18 Is sound a quasiparticle? (Condensed matter)

Sound waves in the harmonic approximation are non-interacting – a general solution is given by a linear combination of the individual frequency modes. Landau’s Fermi liquid theory (footnote 23, page 172) describes how the non-interacting electron approximation can be effective even though electrons are strongly coupled to one another. The quasiparticles are electrons with a screening cloud; they develop long lifetimes near the Fermi energy; they are described as poles of Greens functions.

(a) Do phonons have lifetimes? Do their lifetimes get long as the frequency goes to zero? (Look up ‘ultrasonic attenuation’ and Goldstone’s theorem.)

(b) Are they described as poles of a Green’s function? (See Section 9.3 and Exercise 10.9.)

Are there analogies for phonons to the screening cloud around a quasiparticle? A phonon screening cloud would be some kind of collective, nonlinear movement of atoms that behaved at long distances and low frequencies like an effective, harmonic interaction. In particular, the effective scattering between these quasiphonons should be dramatically reduced from that one would calculate assuming that the phonons are harmonic vibrations.

At low temperatures in perfect crystals (other than solid helium), anharmonic interactions are small except at high sound amplitudes. But as we raise the temperature, anharmonic effects lead to thermal expansion and changes in elastic constants. We routinely model sound in crystals in terms of the density and elastic constants at the current temperature, not in terms of the ‘bare’ phonon normal modes of the zero-temperature crystal.
Day 2

10.11 Onsager regression hypothesis and the harmonic oscillator. ①

A harmonic oscillator with mass $M$, position $Q$ and momentum $P$ has frequency $\omega$. The Hamiltonian for the oscillator is thus $\mathcal{H} = P^2/2M + \frac{1}{2}M\omega^2Q^2$. The correlation function $C_Q(\tau) = \langle Q(t+\tau)Q(t) \rangle$.

(a) What is $C_Q(0)$?  (Hint: Use equipartition.)

The harmonic oscillator is coupled to a heat bath, which provides a dissipative force $-\dot{Q}/\eta$, so under an external force $F(t)$ we have the macroscopic equation of motion $\ddot{Q} = -\omega^2 Q - \dot{Q}/\eta + F/M$ (ignoring thermal fluctuations). We assume the harmonic oscillator is overdamped, so

$$\frac{dQ}{dt} = -\lambda Q + (\lambda/M\omega^2)F,$$

where $\eta = \lambda/M\omega^2$. We first consider the system without an external force.

(b) What is $C_Q(\tau)$, in terms of $\lambda$ and your answer from (a)?  (Hint: Use the Onsager regression hypothesis to derive an equation for $dC_Q/d\tau$.)

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Fig. 1 Two springs. Two harmonic oscillators, with an external force $2F(t)$ coupled to the average displacement $(Q(t) + q(t))/2 = X(t)/2$ (equivalent to $F(t)$ coupled to the sum $Q + q = X$).

Now consider a system of two uncoupled harmonic oscillators, one as described above and another with mass $m$, position $q$, momentum $p$, and frequency $\Omega$, coupled to a heat bath with overdamped decay rate $\Lambda$. (Capital letters are meant to indicate size: this is a small mass with a high frequency and a quick decay time.) Here we only observe $X = Q + q$. Over the following few exercises, we shall compute properties of this combination $X$.

①That is, $Q$ and $q$ represent microscopic degrees of freedom in a material, and $X$ represents a macroscopic property whose correlation function Onsager is trying to estimate.
(c) What is $C_X(0)$? $C_X(\tau)$? Roughly sketch a graph of the latter, assuming $\lambda \approx \Lambda/10$ and $M\omega^2 \approx m\Omega^2/2$. (Hint: They are completely uncoupled, and so also uncorrelated random variables.)

A consequence of the Onsager regression hypothesis is that the correlation function decays ‘according to the same laws as [a deviation] that has been produced artificially’. But clearly there are different ways in which $X(t)$ could evolve from an artificially imposed initial condition, depending on the way we start $Q$ and $q$.

(d) How would $X(t)$ decay from an initial state $Q(0) = X(0)$, $q(0) = 0$? From a state $q(0) = X(0)$, $Q(0) = 0$? Does either decay the way $C_X(t)$ decays from part (c)? How would $X(t)$ decay from an initial state prepared with a sudden impulsive force $F(t) = J_{imp}\delta(t)$? From an initial state prepared with a constant force $F(t) = F_0\Theta(-t)$? Does either decay in the same fashion as $C_X(t)$?

Onsager’s macroscopically perturbed system must be perturbed slowly compared to the internal dynamical relaxation times, for its future evolution to be determined by its current state, and for his hypothesis to be well posed. Stated another way, Onsager hypothesizes that we have included all of the slow variables into the macroscopic observables whose dynamics and correlations are under consideration.

Day 3

10.13 Susceptibility and dissipation for the harmonic oscillator.  

This is a continuation of exercise 10.11, where we considered damped harmonic oscillators. Consider one harmonic oscillator ($P, Q$) with $\mathcal{H} = P^2/2M + \frac{1}{2}M\omega^2Q^2 - F(t)Q(t)$, overdamped with decay rate $\lambda$.

The static susceptibility $\chi_0$ is defined as the ratio of response $Q$ to force $F$.

(a) Under a constant external force $F_0$, what is the equilibrium position $Q_0$? What is the static susceptibility $\chi_0$? (Hint: Ignoring fluctuations, this is just freshman mechanics. What is the spring constant for our harmonic oscillator?) Generally speaking, a susceptibility or compliance will be the inverse of the elastic modulus or stiffness – sometimes a matrix inverse.

Consider our harmonic oscillator held with a constant force $F$ for all $t < 0$, and then released. For $t > 0$, the dynamics is clearly an exponential decay (releasing the overdamped harmonic oscillator), and can also be written as an integral of the time-dependent susceptibility $\chi(\tau)$ for the oscillator:

$$Q(t) = Q_0 \exp(-\lambda t)$$
$$= \int_{-\infty}^{0} dt' \chi(t - t')F$$

(Note that the integral ends at $t' = 0$, where the force is released.)
(b) Solve eqn 2 for $\chi(\tau)$. (Hint: You can take derivatives of both sides and integrate the right-hand-side by parts. Or you can plug in $\chi(\tau)$ as an exponential decay $A \exp(-Bt)$ and solve.)

We can now Fourier transform $\chi$ to find $\tilde{\chi}(\omega) = \int dt \exp(i\omega t) \chi(t) = \chi'(\omega) + i\chi''(\omega)$.

(c) What is $\tilde{\chi}(\omega)$? $\chi'(\omega)$? $\chi''(\omega)$? Is the power dissipated for an arbitrary frequency dependent force $f_\omega$ (given by eqn 10.37 always positive?)