1. **Topological defects in nematic liquid crystals.** (Mathematics, Condensed matter)

   The winding number $S$ of a defect is $\theta_{\text{net}}/2\pi$, where $\theta_{\text{net}}$ is the net angle of rotation that the order parameter makes as you circle the defect. The winding number is positive if the order parameter rotates in the same direction as the traversal (Fig. 1(a)), and negative if it rotates in the opposite directions (Fig. 1(b)).

   ![Fig. 1 Defects in nematic liquid crystals.](image)
   (a) $S = 1/2$ disclination line. (b) $S = -1/2$ disclination. The dots are not physical, but are a guide to help you trace the orientations (starting on the left and moving clockwise); nematic liquid molecules often have a head and tail, but there is no long-range correlation in which direction the heads lie.

   As you can deduce topologically (Fig. 9.5(b) on p. 194), the winding number is *not* a topological invariant in general. It is for superfluids $S^1$ and crystals $T^D$, but not for Heisenberg magnets or nematic liquid crystals (shown). If we treat the plane of the figure as the equator of the hemisphere, you can see that the $S = 1/2$ defect rotates around the sphere around the left half of the equator, and the $S = -1/2$ defect rotates around the right half of the equator. These two paths can be smoothly deformed into one another; the path shown on the order parameter space figure (Fig. 9.5(b)) is about half-way between the two. **Which figure below represents the defect configuration in real space half-way between $S = 1/2$ and $S = -1/2$, corresponding to the intermediate**
path shown in Fig. 9.5(b) on p. 194? (The changing shapes denote rotations into the third dimension.)
Friday
Read: Chapter 10, Sec. 10.1 (Correlation functions: motivation), and 10.2 (Experimental probes of correlations)

2. Human correlations.

Consider a person-to-person correlation function for the horizontal center-of-mass positions \( h_\alpha = (x_\alpha, y_\alpha) \) of people labeled by \( \alpha \), in various public places. In particular, let \( C_{\text{crowd}}(r) = \langle \rho(h) \rho(h + r) \rangle \), where \( \rho(h) = \sum_\alpha \delta(h - h_\alpha) \), and the ensemble average \( \langle \ldots \rangle \) is over a large number of similar events.

Sketch a rough contour plot for \( C_{\text{crowd}}(r) \), for a subway car of length \( L = 10 \text{m} \) with two benches along the edges \( x = 0 \) and \( x = 3 \text{m} \). Assume the people are all seated, with random distances \( \delta y = 0.5 \pm 0.2 \text{m} \) between neighbors.

Monday
Read: Chapter 10, Sec. 10.3 (Equal-time correlations in the ideal gas) and 10.4 (Onsager’s regression hypothesis and time correlations)

3. Liquid free energy.

The free energy \( F \) of a liquid (unlike an ideal gas), resists rapid changes in its density fluctuations \( \rho(x) \):

\[
F(\rho) = \int \frac{1}{2}K(\nabla \rho)^2 + \frac{1}{2}\alpha (\rho - \rho_0)^2 \, dx.
\]

(See Exercise (9.4), where a similar term led to a surface tension for domain walls in magnets.) Thus in a periodic box of length \( L = 1 \),

\[
F(\bar{\rho}) = \sum_m \frac{1}{2}(Kk_m^2 + \alpha)|\bar{\rho}_m|^2 L, \tag{1}
\]

with \( k_m = 2\pi m / L \).

(a) Show that equal-time correlation function is \( \widetilde{C}(k_m) = |\bar{\rho}(k_m)|^2 = k_B T / (Kk_m^2 + \alpha)L \).

(Hint: \( F \) is a sum of harmonic oscillators; use equipartition.) How should we express it in the continuum limit, where \( L \to \infty \)? (Hint: \( \sum_m \equiv L/(2\pi) \int dk \). Alternatively, as a function of continuous \( k \), \( C(k) = k_B T / (Kk^2 + \alpha) \), but remember the \( 1/2\pi \) in the continuum inverse Fourier transform of eqn (A.7).)

(b) Transform to real space. Does the surface tension introduce correlations (as opposed to those in the ideal gas)? Do the correlations fall off to zero at long distances? (Hint: \( \int_{-\infty}^{\infty} \exp(iqy)/(1 + q^2) \, dq = \pi \exp(|y|) \) \[^{1}\] )

The density of a liquid moving through a porous medium will macroscopically satisfy the diffusion equation.

\[^{1}\] The function \( 1/(1 + q^2) \) is often called a Lorentzian or a Cauchy distribution. Its Fourier transform can be done using Cauchy’s residue theorem, using the fact that \( 1/(1 + q^2) = 1/((q + i)(q - i)) \) has a pole at \( q = \pm i \). So \( \int_{-\infty}^{\infty} \exp(iqy)/(1 + q^2) \, dq = \pi \exp(|y|) \). The absolute value happens because one must close the contour differently for \( y > 0 \) and \( y < 0 \).
According to Onsager’s regression hypothesis, what will the correlation $\tilde{C}(k, t)$ be? (See footnote 12 and the new footnote 13.)

It is non-trivial to Fourier transform this back into real space, but the short-distance fluctuations (as they do in the ideal gas) will get smoothed out even more with time.

Exercises

9.4: Domain walls in magnets
9.7: Superfluid order and vortices
10.1: Microwave background radiation
10.5: Telegraph noise in nanojunctions. In part (c), show that

$$P_{\alpha\alpha}(\tau) = \left\langle \frac{(R(t) - R_\beta)(R(t + \tau) - R_\beta)}{(R_\alpha - R_\beta)^2 \rho_\alpha} \right\rangle. \quad (2)$$