Pre-class Preparation

The pre-class question is due 9:30pm on the night before class.

Wednesday
Read: Chapter 2, Sec. 2.3 (Currents and forces)

1. Local conservation.  
   Tin is deposited on a surface of niobium. At high temperatures, the niobium atoms invade into the tin layer. Is the number of niobium atoms $\rho_{\text{Nb}}(x)$ a locally conserved quantity? Politicians in ‘red states’ (primarily Republican) are concerned about migration of democrats into the Sun Belt (the southwest). Is the number of democrats $\rho_{\text{Dem}}(x)$ locally conserved?

Friday
Read: Chapter 2, Sec. 2.4 (Solving: Fourier & Green)

2. Absorbing boundary conditions.  
   A particle starting at $x'$ diffuses on the positive $x$ axis for a time $t$, except that whenever it hits the origin it is absorbed. The resulting probability density gives the Green’s function $\rho(x, t) = G(x|x', t) = \int G(x|x', t)\rho(x', 0)dx'$.

   Solve for $G$. (Hint: Use the ‘method of images’: add a negative $\delta$ function at $-x'$.) How would one use $G$ to evolve an arbitrary initial probability distribution $\rho(x, t_0)$ with this boundary condition?
Monday
Read: Chapter 3, Sec. 3.1 (Microcanonical) and 3.2 (Ideal Gas)

3. **Weirdness in high dimensions.**

We saw in momentum space that most of the surface area of a high-dimensional sphere
is along the equator. Consider the volume of a high-dimensional sphere.

*Is most of the volume near the center, or the surface? How might this relate to statistical
mechanics treatments of momentum space, which approximate the volume of an energy
shell with the volume of the entire sphere?*

**Exercises**

2.5: *Generating random walks (hints on HW web site)*

2.11: *Stocks, volatility, and diversification (hints on HW web site)*

8.4: *Red and green bacteria*, treated as a random walk in the number of red bacteria. Full
credit for sensible arguments that get within a factor of two of the right answer.


A.11: *Random walks, generating functions, and diffusion.*

1. **Random walks, generating functions, and diffusion** (Math)

Consider a one-dimensional random walk with step-size ±1 starting at the origin. What
is the probability \(f_t\) that it first returns to the origin at \(t\) steps?

(a) *Argue that the probability is zero unless \(t = 2m\) is even. How many total paths are
there of length \(2m\)? Calculate the probability for \(f_{2m}\) for up to eight-step hops \((m < 5)\)
by drawing the different paths that touch the origin only at their endpoints.* (Hint: You can save paper by drawing the paths starting to the right, and multiplying by two. Check your answer by comparing to the results for general \(m\) below.)

This *first return* problem is well-studied, and is usually solved using a *generating function*. Generating functions extend a series (here \(f_{2m}\)) into a function. The generating function for our one-dimensional random walk first return problem, it turns out, is

\[
F(x) = \sum_{m=0}^{\infty} f_{2m} x^m = 1 - \sqrt{1 - x} \tag{1}
\]

\[
= \sum_{m} \frac{2^{-2m}}{2m - 1} \binom{2m}{m} x^m.
\]

(look up the derivation if you want an example of how it is done.)

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\[\text{1This problem was created with the assistance of Archishman Raju.}\]
What is the exact probability of returning to the origin for the first time at \( t = 20 \) steps (\( m = 10 \))? Use Stirling’s formula \( n! \sim \sqrt{2\pi n}(n/e)^n \) to give the probability \( f_{2m} \) of first returning at time \( t = 2m \) for large \( t \). (Note that the rate per unit time \( R_{\text{hop}}(t) = f_{2m}/2 \).)

Now let us take the continuum limit for our random walker. Consider a diffusion equation for the probability \( \rho(r, t) \) for a particle starting at position \( \epsilon \) with diffusion constant \( D \) and absorbing boundary conditions at the origin. Let \( P_{\text{cont}} \) be the probability density in time of our particle first touching the origin (and being absorbed) at time \( t \) (like \( P_{\text{hop}} \) but for the continuum limit).

(c) Write a relation between \( P_{\text{cont}}(t) \) and the probability of a random walker surviving to time \( t \) with absorbing boundary conditions. Write a relation between \( P_{\text{cont}}(t) \) and the current of probability at the origin. Finally, write a relation between \( P_{\text{cont}}(t) \) and the slope of the probability density with respect to position at the origin.

You solved for the Green’s function for a diffusion equation with an absorbing boundary condition in a pre-class exercise using the method of images. (Feel free to check it with external sources.)

(d) Calculate \( D \) for the discrete hopping problem of parts (a) and (b), assuming each step takes time \( \Delta t = 1 \). What value of \( \epsilon \) (the initial position of the continuum walker) is needed to make \( P_{\text{cont}} \) correspond correctly to \( P_{\text{hop}} \) at long times \( t \)? (Warning: Remember that \( P_{\text{hop}}(t) \) is zero for odd times \( t \)).

There are lots of other uses for generating functions: analyzing partial sums, finding moments, taking convolutions...

(e) Argue that \( F(1) \) is the probability that a particle will eventually return to the origin, and that \( F'(1) \) is \( \langle m \rangle \), half the expected time to return to the origin. What is the probability that our one-dimensional walk will return to the origin? What is the mean time to return to the origin?

A.15 (Extra credit): Diffusion equation and universal scaling functions. (This exercise covers material in Chapter 12, which is largely independent of the other parts of the book, but is very sophisticated. To do the problem, you will at least need to read some of that chapter to find out what ‘relevant’, ‘marginal’, and ‘irrelevant’ mean. Read more of the chapter to find out why universal scaling functions and power laws are important.)

2. Diffusion equation and universal scaling functions

The diffusion equation universally describes microscopic hopping systems at long length scales. We will investigate how to write the evolution in a universal scaling form.

The solution to a diffusion problem with a non-zero drift velocity is given by \( \rho(x, t) = 1/\sqrt{4\pi Dt} \exp(-((x-\epsilon t)^2/(4Dt))) \). We will coarse grain by throwing away half the time points. We will then rescale the distribution so it looks like the original distribution.

\(^2\)This exercise was developed in collaboration with Archishman Raju.
We can just write these two operations as $t' = t/2$, $x' = x/\sqrt{2}$, $\rho' = \sqrt{2}\rho^3$. These three together constitute our renormalization group operation.

(a) Write an expression for $\rho'(x', t')$ in terms of $D$, $v$, $x'$, and $t'$ (not in terms of $D'$ and $v'$). Use it to determine the new renormalized velocity $v'$ and diffusion constant $D'$. Are $v$ and $D$ relevant, irrelevant or marginal variables?

Typically, whenever writing properties in a scaling function, there is some freedom in deciding which invariant combinations to use. Here let us use the invariant combination of variables, $X = x/\sqrt{t}$ and $V = \sqrt{tv}$. We can then write

$$\rho(x, t) = t^{-\alpha}P(X, V, D),$$

a power law times a universal scaling function of invariant combination of variables.

(b) Show that $X$ and $V$ are invariant under our renormalization group operation. What is $\alpha$? Write an expression for $P$, in terms of $X$, $V$, and $D$ (and not $x$, $v$, or $t$).

(Note that we need to solve the diffusion equation to find the universal scaling function $P$, but we can learn a lot from just knowing that it is a fixed point of the renormalization group. So, the universal exponent $\alpha$ and the invariant scaling combinations $X$, $V$, and $D$ are determined just by the coarsening and rescaling steps in the renormalization group. In experiments and simulations, one often uses data to extract the universal critical exponents and universal scaling functions, relying on emergent scale invariance to tell us that a scaling form like eqn 2 is expected.)

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3Because $\rho$ is a density, we need to rescale $\rho' dx' = \rho dx$. 