Physics 218: Waves and Thermodynamics
Fall 2002, James P. Sethna
Homework 5, due Monday Sept. 30
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Computer Labs

Fourier Series and Transforms, Monday evening 9/23 and Thursday afternoon 9/26, Rock B3 (hidden around the corner in the basement).

Prelim I

Prelim I is scheduled in two weeks, Monday October 7 (subject to discussion in class). The content will focus on the homework, with some questions from the experimental lab Standing Waves and the two Fourier labs. There will be several multiple-choice questions (no partial credit) and one or perhaps two longer multiple-part essay questions. Next Monday, instead of a problem set, I will pass out copies of last year’s Prelim I for you to use while studying.

Reading

Elmore & Heald, sections 5.1-5.3, 5.10, 12.3/5
Feynman, section I.48-5/6, I.49-3/5, I.50-5/6, I.51 1/2, I.52

(5.1) Translation of Problem Set 1 into Modern Language.

The first two questions assigned in problem set 1 are roughly equivalent to the following question, posed in a form that would be appropriate in a graduate physics special topics course:

“Incorporate into the wave equation the leading order terms breaking time-reversal invariance and invariance under changing the sign of the order parameter. Give a possible physical origin for each term.”

You’ve already solved this problem: we just want you to translate the question into the language of modern condensed-matter physics (as we began to discuss in class). (a) Which term, gravity or friction, breaks time-reversal invariance? (b) Which term breaks invariance under changing the sign of the order parameter?

(5.2) Decibels. Look up the decibel scale on the Web. The threshold of hearing is around zero decibels (0 dB). From this and your knowledge of air and sound, estimate the amplitude of the vibration of your eardrum at the threshold of audibility. (The bulk modulus of air $B$ is about $1.4 \times 10^5 \, N/m^2$; the density of air is about $1.2 \, kg/m^3$; the speed of sound in air is about $340 \, m/s$; a typical sound frequency might be $1000 \, Hz$.) Compare this with other natural scales of length: which is it closest to, the size of your ear, the width of a hair in your cochlea, the width of a cell, the width of an atom, ....
An array of pendula connected by springs, in the continuum limit, obeys the Sine-Gordon equation

\[ \frac{\partial^2 \phi}{\partial t^2} = A \frac{\partial^2 \phi}{\partial x^2} - B \sin(\phi). \]

with \( \phi(x) = 0 \) corresponds to the pendulum at position \( x \) along the array pointing downward. What is the dispersion relation \( \omega(k) \) for small oscillations in this equation?

(A) \( \omega(k) = \left( -B \pm \sqrt{B^2 - 4Ak^2} \right) / 2A \)
(B) \( \omega(k) = \sqrt{Ak^2 - B \sin(\phi)} \)
(C) \( \omega(k) = \sqrt{Ak^2 - B} \)
(D) \( \omega(k) = \sqrt{Ak^2 + B} \)
(E) \( \omega(k) = \sqrt{Ak^2 + B \sin(\phi)} \)

(5.4) Deriving New Laws.
The evolution of a physical system is described by a field \( \Xi \), obeying a partial differential equation

\[ \frac{\partial \Xi}{\partial t} = A \frac{\partial \Xi}{\partial x}. \] (S3.1)

(a) Symmetries.
Give the letters corresponding to ALL the symmetries that this physical system appears to have:

(A) Spatial inversion \( (x \rightarrow -x) \).
(B) Time reversal symmetry \( (t \rightarrow -t) \).
(C) Order parameter inversion \( \Xi \rightarrow -\Xi \).
(D) Homogeneity in space \( (x \rightarrow x + \Delta) \).
(E) Time translational invariance \( (t \rightarrow t + \Delta) \).
(F) Order parameter shift invariance \( (\Xi \rightarrow \Xi + \Delta) \).

(b) Traveling Waves. Show that our equation \( \partial \Xi / \partial t = A \partial \Xi / \partial x \) has a traveling wave solution. If \( A > 0 \), which directions can the waves move?
A sound wave generator generates a triangular pressure air wave moving toward the right down a hollow tube, as shown in the figure above. The triangles repeat forever with wavelength $L$. The maximum displacement of the wave is $A$, the velocity of sound is $v$, and the bulk modulus for air is $B$.

(a) What is the intensity (power per unit area) traveling down the tube?

The figure shows the Fourier series for our wave truncated at $n = \pm 2$ and $n = \pm 4$.

(b) We now want to decompose this intensity into different frequencies. What would the time average $I_{av}^n$ for the intensity of a single traveling plane wave of wave vector $k_n$ and amplitude $a_n$, $u_n(x, t) = a_n \sin(k_n(x - vt))$? (Leave your answer in terms of $a_n$ and $k_n$.)

The Fourier series for the displacement of the wave is

$$u(x) = \sum_{n=0}^{\infty} a_n \sin(k_n(x - vt))$$

with $k_n = 2\pi n/L$. The Fourier coefficients are $a_n = 0$ for $n$ even, and

$$a_n = (-1)^{(n-1)/2}A/(\pi^2n^2)$$

for $n$ odd.

(c) Verify explicitly that the sum of the intensities per frequency channel $n$ you calculated in part (b) equals the total intensity you calculated in part (a). You’ll need the formula

$$\pi^2/8 = 1 + 1/3^2 + 1/5^2 + 1/7^2 + \ldots$$

This is Feynman’s energy theorem, section I.50-5: the energy of the sum of different Fourier waves is the sum of the energies of the individual waves. This is why we can talk about the power spectrum of a wave: you can think of the power at different frequencies as being independent of one another.
(5.6) Pythag: Group velocity, phase velocity, and dispersion.

Start up Pythag. Choose Packet forcing on the left-hand side: this yanks on the left with an amplitude given by a Gaussian pulse of FWHM 0.015 seconds times a sinusoidal modulation of frequency $\Omega = 300$ radians per second. Hit Initialize and Run, and watch the packet bounce back and forth. As is usual with the wave equation, the pulse propagates without changing in shape. This is only true, however, so long as the pulse does not change much on the length scale given by the distance between points $\delta x$ on the numerical string.

Open the Configure menu. Change $\Omega$ to 800 and $FWHM$ to 0.015. To slow down the pulse, change graph time skip to 1. You should now see a pulse which changes shape as it moves.

(a) Is the group velocity faster or slower than the phase velocity? This is easiest to see by looking at the pulse early on, before it stretches out: do the peaks within the wave of the carrier frequency move forward faster or slower than the pulse as a whole?

After several passes across the window, you should see a broad pulse, which has longer waves on one side than the other.

(b) Does the leading edge have longer or shorter wavelength than the trailing portion of the packet? Which wavelengths move faster, the long wavelengths or the short ones?

(c) Do these two answers agree with what you found for the dispersion relation in problem set 4?

Now change the number of string pieces (chunks) to 999 (the largest value allowed), and change the graph time skip back to 20.

(d) Does the dispersion go away when you reduce the spacing $\delta x$ in this way?