The multiple choice problems have been designed so that, if you are inspired, you can do them quickly without calculations. There are no tricks to speed up the short answer question.

Multiple Choice (48 pts): ____ x 12pts = ____________

Short Answer: S1 (52 pts) ________________

TOTAL ________________
Multiple Choice: Be sure to put answers in boxes provided. (Sorry: no partial credit!)

M1. (12 pts) Reflection, Transmission, and Boundary Conditions.

The string shown above has speed of sound $c_1 = 5\text{m/s}$ on the left and $c_2 = 10\text{m/s}$ on the right. The boundary conditions on the right end are fixed at $L = 20\text{m}$; the left end is at $x = 0\text{m}$, and the massless knot connecting the two strings is halfway, at $x = 10\text{m}$. A pulse at $t = 0$ centered at $x_0 = 5\text{m}$ is traveling to the right. The transmitted pulse is observed to hit the end of the string ($x = L$) at about 2 seconds. Which figure below correctly represents the pulse at time $t = 2.4$ seconds? (All graphs are on the same scale.)

(A) ![Graph A]

(B) ![Graph B]

(C) ![Graph C]

(D) ![Graph D]

(E) ![Graph E]

Answer


M2. (12 pts) Fourier Series and Uncertainty.

The dashed line above shows $G_0(x) = 1/\sqrt{2\pi} \exp(-x^2/2)$. The dark line shows another function $G(x)$. The areas under the two curves $G(x)$ and $G_0(x)$ are the same. The dashed lines in the choices below represent the Fourier transform $\tilde{G}_0(k) = \exp(-k^2/2)$. Which solid curve below represents the Fourier transform of the black curve above?

(A)  
(B)  
(C)  
(D)  
(E)  

Answer

Beware: the Fourier lab program auto-rescaled the height.

On the left is the current shape $\eta(x,0)$ of a packet at time $t = 0$, traveling to the left on a string of mass density $\lambda_0 = 0.1 \text{kg/m}$ and tension $\tau = 160 \text{N}$. What is the quantity shown on the graph at right?

(A) Chunk velocity $\partial \eta / \partial t$.
(B) Total energy density $1/2 \lambda_0 (\partial \eta / \partial t)^2 + 1/2 \tau (\partial \eta / \partial x)^2$.
(C) Power $-\tau \partial \eta / \partial x \partial \eta / \partial t$.
(D) Momentum $-\lambda_0 \partial \eta / \partial x \partial \eta / \partial t$.
(E) Time independent, $\tilde{\eta}(x,t) = \eta(x,t - \Delta)$.
(F) Sideways motion, $\tilde{\eta}(x,t) = \eta(x,t) + \Delta$.

Answer

Hint: Which quantities are negative for a pulse moving left?
M4. (12 pts) Group and Phase Velocities.

An unusual crystal has a dispersion relation \( \omega[k] = A\sqrt{k} \) (left figure above). It is vibrating with a superposition of two rightward-moving sinusoidal traveling waves with nearly the same wavevector \( k \pm \delta k \):

\[
\eta(x, t) = \cos((k + \delta k)x - \omega[k + \delta k]t) + \cos((k - \delta k)x - \omega[k - \delta k]t).
\]

At \( t = 0 \) the superposition has the form shown above on the right: it consists of an envelope wave traveling to the right at a velocity \( v_E \) and a carrier wave traveling to the right at velocity \( v_C \). Which of the following gives the relation of \( v_E \) to \( v_C \) for this system? (You may ignore corrections of order \((\delta k)^2\).

(A) \( v_E = v_C \) as \( k \to 0 \)
(B) \( v_E = 2v_C \)
(C) \( v_E = v_C/2 \)
(D) \( v_E[k] \sim \sqrt{2 - 2\cos(ka)} \) and \( v_C[k] \sim -\sqrt{2 - 2\cos(ka)} \)
(E) \( \partial^2 v_E/\partial t^2 = c^2 \partial^2 v_C/\partial x^2 \).

Answer

\( \square \)
Short Answer: Show Your Work

S1. (52 pts) Wave Equation under Gravity.

A string of mass density $\lambda_0$ dangles from the ceiling. Height $h$ is measured from the bottom of the string: the horizontal displacement is $\eta(h)$ (above).

Gravity acting on the string breaks two symmetries of the ordinary wave equation. First, the string no longer has reflection symmetry along its length: under the change $h \rightarrow -h$ the top and bottom of the string are interchanged. Second, the string is no longer homogeneous along its length: the upper parts of the string are under higher tension than the lower parts (since the former need to support the weight of the latter).

(A) **Broken Symmetries.** (12 points)

Which of the following constants (A,B,C,D,E,F) that were not allowed in the ordinary wave equation are now allowed to be non-zero by symmetry?

\[
\frac{\partial^2 \eta}{\partial t^2} = A + Bt + C\eta + D\frac{\partial \eta}{\partial h} + E\frac{\partial \eta}{\partial t} + (c^2 + Fh)\frac{\partial^2 \eta}{\partial h^2}.
\]  

(S.1)
The free body diagram for a chunk of the dangled string is shown above. We assume that the chunk motion is purely transverse (in the $x$ direction) so the vertical forces must add to zero, as shown. Give the formulas for the horizontal components of the forces on the two ends of the chunk, in terms of $\lambda_0$, $g$, $h$, $\delta h$, $\theta_1$, and $\theta_2$.

$$F_x[h] =$$

$$F_x[h + \delta h] =$$

Did you leave your answer in terms of $\theta_1$ and $\theta_2$? Wait until part (C) to convert to derivatives.
(C) (20 points)

Write the equation of motion for the dangling string, by setting the total horizontal force from part (B) equal to $ma$ for the chunk. Write your answer in terms of $\lambda_0$, $g$, $h$, $\partial \eta / \partial h$, and $\partial^2 \eta / \partial h^2$; it should not involve $\delta h$, $\theta_1$, or $\theta_2$.

\[
\frac{\partial^2 \eta}{\partial t^2} = \quad \\
\text{Hint: your answer might involve terms that are allowed by symmetry in part (A).}
\]
Physics 218—Exam I (October 7) Fall 2002

Formula Sheet
James P. Sethna

Complex Trigonometry. $\exp(iz) = \cos(z) + i\sin(z)$, $\cos(z) = (\exp(iz) + \exp(-iz))/2$, and $\sin(z) = (\exp(iz) - \exp(-iz))/(2i)$. $\sin(A + B) = \sin A \cos B + \cos A \sin B$, $\cos(A + B) = \cos A \cos B - \sin A \sin B$, $\exp(iA) + \exp(iB) = \exp((A + B)/2)2\cos((A - B)/2)$, $\cos(A) + \cos(B) = 2\cos((A + B)/2)\cos((A - B)/2)$.

Fourier. 
\[ \hat{y}_m = (1/L) \int_0^L y(x) \exp(-ik_m x) dx \text{ where } k_m = 2\pi m/L; \ y(x) = \sum_{m=-\infty}^{\infty} \hat{y}_m \exp(ik_m x). \]
\[ \hat{y}(k) = \int_{-\infty}^{\infty} y(x) \exp(-ikx) dx; \ y(x) = (1/2\pi) \int_{-\infty}^{\infty} \hat{y}(k) \exp(ikx) dk. \]
\[ \hat{y}_m^{FFT} = \sum_{\ell=0}^{N-1} y_\ell \exp(-i2\pi m\ell/N) \text{ with } m = 0, \ldots, N-1; \ \hat{y}_m^{FFT} = \sum_{\ell=0}^{N-1} y_\ell \exp(-ik_m x_\ell). \]
\[ G(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right); \ \tilde{G}(k) = \exp(-ikx_0)\exp(-\sigma^2k^2/2). \]

Orthonormality.
\[ (1/L) \int_0^L \exp(ik_m x) \exp(ik_n x) dx = \delta_{mn}, \text{ where } \delta_{mn} = 0 \text{ for } m \neq n \text{ and } \delta_{mn} = 1 \text{ for } m = n. \]

The Fourier transform of a Gaussian of width $\sigma$, $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-x^2/2\sigma^2\right)$ is a Gaussian $\hat{f}(k) = \exp\left(-\sigma^2k^2/2\right)$ of width $1/\sigma$. The Fourier transform of $f(x-x_0)$ is $\exp(-ikx_0)\hat{f}(k)$.

Wave Equation $\partial^2 \eta / \partial t^2 = c^2 \partial^2 \eta / \partial x^2$. Traveling wave solution $\eta(x, t) = f(x \pm ct)$. Standing-wave solution $\eta(x, t) = A \sin(kx) \sin(\omega t)$. Traveling sine wave (Plane wave) $\eta(x, t) = A \sin(kx - \omega t)$, with $\omega/k = c$.

\[ c = \sqrt{\tau/\lambda_0}, \ K(x, t) = (\lambda_0/2)(\partial\eta/\partial t)^2, \ V(x, t) = (\tau/2)(\partial\eta/\partial x)^2, \ E(x, t) = K(x, t) + V(x, t), \ P(x, t) = -\tau(\partial\eta/\partial t)(\partial\eta/\partial x) g_x(x, t) = -\lambda_0(\partial\eta/\partial t)(\partial\eta/\partial x). \]

Traveling waves, $\eta(x, t) = f(x \pm ct)$ have special properties: $\partial\eta/\partial x = \pm(1/c)(\partial\eta/\partial t)$, $K(x, t) = V(x, t), \ E(x, t) = \tau(\partial\eta/\partial x)^2$.

Sound: $c = \sqrt{B/\rho} = 340 \text{ m/s for air.} \ p = -B(\partial s/\partial x), \ \partial^2 s/\partial t^2 = -(1/\rho)(\partial p/\partial x)$.

General Solutions. Bounded at infinity.

Finite string: $\eta(x, t) = \sum_{n=1}^{\infty} \sin(k_n x)(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)), \ \omega_n = ck_n, \ k_n = n\pi/L$. Infinite string: $\eta(x, t) = f(x - ct) + g(x + ct)$; initial displacement $f(x) + g(x)$, initial velocity $c(g'(x) - f'(x))$.

Boundary Conditions. Fixed: $\eta = \partial\eta/\partial t = 0$. Free: $\partial\eta/\partial x = 0$. Both fixed or free: $\omega_n = cn\pi/L$; mixed $\omega_n = c(2n+1)\pi/2L$.

Symmetries. The wave equation is symmetric under (a) Reflection along $x$, $\tilde{\eta}(x, t) = \eta(-x, t)$, (b) Time reversal, $\tilde{\eta}(x, t) = \eta(x, -t)$, (c) Reflection along $y$, $\tilde{\eta}(x, t) = -\eta(x, t)$. It is also (d) Homogeneous, $\tilde{\eta}(x, t) = \eta(x - \Delta, t)$, (e) Time independent, $\tilde{\eta}(x, t) = \eta(x, t - \Delta)$, and invariant under (f) Sideways motion, $\tilde{\eta}(x, t) = \eta(x, t) + \Delta$.

Reflection and Transmission (Massless Knot). $R = (Z_1 - Z_3)/(Z_1 + Z_2)$ and $T = 2Z_1/(Z_1 + Z_2)$ where impedances $Z_i = \sqrt{\lambda_i/\tau_i}$. If the tension is constant, $R = (c_2 - c_1)/(c_2 + c_1)$ and $T = 2c_2/(c_2 + c_1)$

Group and Phase Velocities. Phase velocity $\omega(k)/k$. Group velocity $d\omega/dk$. One dimensional chain $\omega(k) = \sqrt{K/M}\sqrt{2 - 2\cos(ka)}$. 9
The multiple choice problems have been designed so that, if you are inspired, you can do them quickly without calculations. There are no tricks to speed up the short answer question.

Multiple Choice (40 pts):____ x 10 pts = ____________

Short Answer: S1 (60 pts) ________________

TOTAL ________________
Multiple Choice: Be sure to put answers in boxes provided. (Sorry: no partial credit!)

M1. (10 pts) The Squid and the Bathysphere.

A scientist is exploring the bottom of the sea in a bathysphere, where the weight of the water above exerts a large, uniform pressure $P$. A giant squid attacks! It squeezes the diving vessel in an irregular fashion. The normal vector to a point $r$ on the surface of the bathysphere is $\hat{n}$. Assuming that the squeezing is slow (so that the material is in local equilibrium), and assuming that the squid is not directly pressing on the point $r$, what can you say about the stress field $\sigma_{ij}(r)$?

(A) $\sigma_{ij}\hat{n}_i = 0$
(B) $\sigma_{ij}\hat{n}_j = 0$
(C) $\sigma_{ij}\hat{n}_j = -P\hat{n}_i$
(D) $\sigma_{ij}\hat{n}_j = -P\hat{n}_j$
(E) $\mu\sigma_{ij}\sigma_{ij} + (\lambda/2)\sigma_{ii}^2 = P$

Answer
A radio station south of Ithaca and west of Binghamton transmits with a carrier wave of wavelength $\lambda$. They use three antennas spaced equally along a north-south direction, separated by a distance $d = 2/3 \lambda$. The antennas are close together compared to the distance to either city. The radio station cleverly delays the signal in the antennas, so as to add a phase shift $\phi = 2\pi/3$ in the carrier wave between neighboring antennas. In particular, if we number the three antennas from south to north as $\{−1, 0, 1\}$, the radio signal is given by

$$
A(r_0) \left( \exp[i(\omega t - kr_1 + \phi)] + \exp[i(\omega t - kr_0)] + \exp[i(\omega t - kr_{-1} - \phi)] \right)
$$

What is the ratio of the average intensity $I_{av}$ of the radio signal in Ithaca, compared to $I_0$, which one would find with only one antenna transmitting? In Binghamton?

(A) Ithaca: $I_{av}/I_0 = 9$; Binghamton $I_{av}/I_0 = 3$.
(B) Ithaca: $I_{av}/I_0 = 9$; Binghamton $I_{av}/I_0 = 1$.
(C) Ithaca: $I_{av}/I_0 = 9$; Binghamton $I_{av}/I_0 = 0$.
(D) Ithaca: $I_{av}/I_0 = 3$; Binghamton $I_{av}/I_0 = 1$.
(E) Ithaca: $I_{av}/I_0 = 3$; Binghamton $I_{av}/I_0 = 0$.

Answer
M3. (10 pts) Diffraction and Fourier Optics.

A glass slide is coated with soot, blocking all light except for a thin stripe across the center where the soot has been rubbed off. A laser beam of wavelength $\lambda$ is aimed at the glass: the beam width is large compared to the slit width. A careful measurement of the light transmission immediately outside the glass shows that amplitude of the light has a Gaussian profile: it varies with $x$ as $\exp(-2x^2/a^2)$, where $a$ is a width of the transparent strip and $x$ is the distance from the center of the strip. Which of the following intensity patterns will be observed on a distant screen? (You may assume small angles.)

(A) [Image of intensity pattern A]

(B) [Image of intensity pattern B]

(C) [Image of intensity pattern C]

(D) [Image of intensity pattern D]

(E) [Image of intensity pattern E]

Answer: [Choose the correct answer]
M4. (10 pts) Tensors.

In the derivation which gave the wave equation for longitudinal sound waves, we proved the identity
\[ \nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}. \]

Which of the following formulas is the same equality, but in modern tensor notation? (You don’t need to derive the formula, only translate it.)

(A) \( \partial_i \partial_j u_k = \partial_i \epsilon_{ijk} \partial_j u_k - \partial_i^2 u_j \)

(B) \( \epsilon_{ijk} \partial_i \epsilon_{jkl} \partial_l u_m = \partial_m \partial_m u_j - \partial_m \partial_j u_m \)

(C) \( \epsilon_{ijk} \partial_j \epsilon_{k\ell m} \partial_\ell u_m = \partial_i \partial_m u_m - \partial_j \partial_j u_i \)

(D) \( \partial_i \epsilon_{jke} \partial_m \epsilon_{nop} = \partial_i \partial_j u_k - \partial_i^2 u_j \)

(E) \( \partial_i \epsilon_{ijk} \partial_j u_k = \partial_m \partial_m u_j - \partial_m \partial_j u_m \)

Answer
S1. (60 pts) Torsional Wave Equation.

A long thin rod is twisted about its long axis by an angle $\theta(x) = A \cos(kx - \omega t)$, where $x$ measures the distance along the rod and $\theta$ measures the angle with respect to the $\hat{z}$ axis, so $u(x, y, z, t) = (0, y(\cos \theta - 1) - z \sin \theta, z(\cos \theta - 1) + y \sin \theta) \approx (0, -z \theta, y \theta)$ where the last approximation is true for small wave amplitudes $A$. In this problem you may assume that $A$ is small, so in particular

$$u(x, y, z, t) = (0, -z A \cos(kx - \omega t), y A \cos(kx - \omega t)).$$

(a) Write the strain tensor field, $\epsilon_{ij}(x, y, z, t)$, out as a $3 \times 3$ matrix. Ignore the geometric nonlinearity, so $\epsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2$.

$$\epsilon_{ij}(x, y, z, t) = \begin{pmatrix}
0 & 0 & -z A \\
0 & y A & 0 \\
-z A & 0 & 0
\end{pmatrix}$$

* You may use a complex wave $\theta(x) = Ae^{i(kx - \omega t)}$ if you prefer.
(b) The torsional rod is made of an isotropic material, with elastic constants $\lambda$ and $\mu$. From your answer for part (a), compute the stress tensor $\sigma_{ij}(x, y, z, t) = 2\mu \epsilon_{ij} + \lambda \epsilon_{kk} \delta_{ij}$. 

\[
\sigma_{ij}(x, y, z, t) = \begin{pmatrix}
\end{pmatrix}
\]
(c) Calculate the force $F_i = \partial_j \sigma_{ij} dV$ on a small volume $dV$ of material at $x, y, z$. 

$$F(x, y, z, t) = \left( \begin{array}{c} \\ \end{array} \right) dV$$
(d) Is the displacement field

\[ \mathbf{u}(x, y, z, t) = (0, -zA \cos(kx - \omega t), yA \cos(kx - \omega t)). \]

curl free or divergence free? Should the wave speed \( c = \omega/k \) be the longitudinal speed of sound \( c_L = \sqrt{(\lambda + 2\mu)/\rho} \) or the transverse speed of sound \( c_T = \sqrt{\mu/\rho} \)?

Circle one of each (possibly one on each side).

Curl Free \hspace{2cm} \text{Divergence Free}

\[ c_L = \sqrt{(\lambda + 2\mu)/\rho} \hspace{2cm} c_T = \sqrt{\mu/\rho} \]
(e) Give the formulas for the force $\mathbf{F}$ (transcribed from part (c)), the mass $m$, and the accelleration $\mathbf{a}$ for a small chunk of rod of volume $dV$. The torsional rod has mass $\rho$ per unit volume. The accelleration should be calculated directly from the displacement field $\mathbf{u}(x, y, z, t) = (0, -zA \cos(kx - \omega t), yA \cos(kx - \omega t))$. Show that $\mathbf{u}$ satisfies Newton’s law $\mathbf{F} = ma$, and solve for the dispersion relation $\omega(k)$. (Show your work: using the answer from part (d) will not suffice.)

\[
\mathbf{F} = \begin{pmatrix} \quad \end{pmatrix}
\]

\[
m = \begin{pmatrix} \quad \end{pmatrix}
\]

\[
\mathbf{a} = \begin{pmatrix} \quad \end{pmatrix}
\]

\[
\omega(k) = \_\_\_\_\_\_\_\_\_\_\_\_.\]
Sound Waves in Three Dimensions.
\[ \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = -\nabla p, \quad p = -B \nabla \cdot \mathbf{u}, \quad \frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p \] with \( c = \sqrt{B/\rho} \), \( \frac{\partial^2 \mathbf{u}}{\partial t^2} = c^2 \nabla^2 \mathbf{u} \).

Spherical waves: \( p(\mathbf{r}, t) = f(|\mathbf{r}| - ct)/|\mathbf{r}| \).

Snell’s law: \( n_1 \sin \theta_1 = n_2 \sin \theta_2 \), where the index of refraction \( n = \sqrt{\epsilon \mu} \) is \( c/v \).

Phase shift \( \pi \) on reflection where the index of refraction increases (e.g., light off glass).

Intensity along the direction of propagation \( I = \rho \partial \mathbf{u} / \partial t \),

energy density \( E = (\rho/2)(\partial \mathbf{u} / \partial t)^2 + p^2/(2B) \).

If \( p(t) = \sum_n \tilde{p}_n \exp(i\omega_n t) \) and \( \rho(t) = \sum_m \tilde{\rho}_m \exp(i\omega_m t) \), then the total power is the sum of the power in each frequency channel: \( \sum_n (-i\omega/2)\tilde{p}_n \tilde{\rho}_n^* \).

Interference and Diffraction.

Double Slit. Phase difference \( \phi = 2\pi d \sin(\theta)/\lambda = kd \sin(\theta) \).

Intensity \( I_{av} = 4I_0 \cos^2(\phi/2) = 4I_0 \cos^2(kd \sin \theta/2) \) (\( I_0 \) single slit intensity).

Constructive for \( d \sin \theta = 0, \pm \lambda, \pm 2\lambda, \ldots \), destructive for \( d \sin \theta = \pm \lambda/2, \pm 3\lambda/2, \ldots \).

Multiple slits. \( I_{av} = I_0 \sin^2(N\phi/2)/\sin^2(\phi/2) \); principle maxima at \( \phi = 0, 2\pi, 4\pi \), destructive at \( \phi = 2m\pi/N \) with \( m \) any integer except \( 0, \pm N, \pm 2N, \ldots \).

Diffraction. If the slit opening is \( f(x), \) \( I_{av} \propto |\tilde{f}(k \sin \theta)|^2 \).

The Fourier transform of a shifted function \( f(x - \Delta) \) is \( \exp(-i\Delta k)\tilde{f}(k) \).

Single wide slit. \( I_{av} = I_{center} \sin^2 \alpha/\alpha^2 \) with \( \alpha = ak \sin(\theta)/2. \)

Tensor Notation.

Einstein convention: \( a_{ijkl} b_{imno} = \sum_{i=1}^{3} a_{ijkl} b_{imno} \).

Dot product \( \mathbf{a} \cdot \mathbf{b} = a_i b_i \), matrix applied to vector \( (M \mathbf{x})_i = M_{ij} x_j \), matrix multiplication \( (MN)_{ij} = M_{ik} N_{kj} \), trace \( \text{Tr}(M) = M_{ii} \).

Laplacian \( \nabla^2 f = \partial_i \partial_i f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f \), divergence \( \nabla \cdot \mathbf{v} = \partial_i v_i \).

Identity tensor \( \delta_{ij} \), equals one if \( i = j \), zero otherwise.

Totally antisymmetric tensor \( \epsilon_{ijk} : \epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = \epsilon_{kji} \).

\( \epsilon_{123} = 1 = \epsilon_{231} = \epsilon_{312}, \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1 \), zero if any index repeats.

\( \mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_j b_k \), \( (\nabla \times \mathbf{v})_i = \epsilon_{ijk} \partial_j v_k \), \( \det M = \epsilon_{ijk} \epsilon_{\ell m n} M_{i\ell} M_{j m} M_{k n} \).

\( \delta_{ii} = 3, \epsilon_{ijk} \delta_{jk} = 0, \epsilon_{ij} \epsilon_{i j} = 6, \epsilon_{ijk} \epsilon_{i j} = 2\delta_{kk}, \epsilon_{ijm} \epsilon_{km} = \delta_{ik} \delta_{j} - \delta_{ij} \delta_{k} \).

Elasticity Theory.

Stress tensor \( \sigma_{ij} \hat{n}_j = \text{Force/Area across surface perpendicular to } \hat{n} \).

\( \sigma_{ij} = \sigma_{ji} \) because torques on small volumes must vanish. Force on a small volume \( F_i = \partial_j \sigma_{ij} \). For hydrostatic pressure \( P \), \( \sigma_{ij} = -P \delta_{ij} \).

Strain tensor \( \epsilon_{ij} = (1/2) (\partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k) \), where the last term (the geometric nonlinearity) is usually ignored. \( \epsilon_{ij} = \epsilon_{ji}. \) The strain tensor for uniform stretching \( -\Delta V/V = 3\Delta L/L \) would be \( \epsilon_{ij} = (\Delta L/L) \delta_{ij} \).
Tensor of elasticity $c_{ijkl}$ gives Hooke’s law for anisotropic media, $\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$. $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$. There are 21 possible independent elastic constants.

The elastic energy density $E = (1/2)\sigma_{ij}\varepsilon_{ij} = (1/2)c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$.

**Isotropic moduli.** The bulk modulus $K$ is the same as $B$ for fluids: $P = -K(\Delta V/V)$.

Under a shear by an angle $\theta$, $E = (1/2)\mu\theta^2$.

Under unconstrained stretching, $F/A = Y\Delta L/L$, and $\Delta W/W = -\nu\Delta L/L$, where $\nu$ is Poisson’s ratio. $K = 2\mu/3 + \lambda$, $\nu = \lambda/(2(\mu + \lambda))$, and $Y = (2\mu^2 + 3\lambda\mu)/(\mu + \lambda)$.

**Isotropic Tensors.** $c_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}$. $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$. $E = \mu\varepsilon_{ij}\varepsilon_{ij} + (\lambda/2)(\varepsilon_{kk})^2$.

**Wave equations.** $\rho_0 \partial^2 u_i/\partial t^2 = \partial_j\sigma_{ij} = (1/2)c_{ijkl}\partial_j(\partial_ku_l + \partial_lu_k)$. For isotropic media, $\rho_0 \partial^2 u_i/\partial t^2 = (\lambda + \mu)\partial_i\partial_ju_j + \mu\partial_j\partial_iu_i$, or $\rho_0 \partial^2 u_i/\partial t^2 = (\lambda + \mu)\nabla(\nabla \cdot u) + \mu\nabla^2 u$.

Decomposing $u = u_T + u_L$ with $\nabla \cdot u_T = 0$ and $\nabla \times u_L = 0$, we have $\partial^2 u_T/\partial t^2 = c_T^2\nabla^2 u_T$ and $\partial^2 u_L/\partial t^2 = c_L^2\nabla^2 u_L$, with $c_T = \sqrt{\mu/\rho_0}$ and $c_L = \sqrt{(\lambda + 2\mu)/\rho}$.

**WKB.**

If $c(x)$ varies slowly, the solution to a wave equation $\partial^2 \eta/\partial t^2 = c(x)^2\partial^2 \eta/\partial x^2$ is approximately of the WKB form $\eta(x,t) = A(x)\cos((S(x) - \omega t))$, where the phase $dS(x)/dx = \omega/c(x) = k(x)$ and the amplitude $A(x) = A(x_0)\sqrt{c(x)/c(x_0)}$.

**Formulas from Prelim I.**

**Trigonometry** $f = \omega/2\pi$, and $k = 2\pi/\lambda$. $\exp(iz) = \cos(z) + i\sin(z)$, $\cos(z) = (\exp(iz) + \exp(-iz))/2$, and $\sin(z) = (\exp(iz) - \exp(-iz))/(2i)$.

**Wave Equation Solutions.** The wave equation

$$\partial^2 \eta/\partial t^2 = c^2\partial^2 \eta/\partial x^2$$

has a traveling wave solution $\eta(x,t) = f(x \pm ct)$, a standing-wave solution $\eta(x,t) = A\sin(kx)\sin(\omega t)$, and (as a special case) a traveling sine wave $\eta(x,t) = A\exp(i(kx - \omega t))$, where $\omega/k = c$.

**Fourier Transform of a Gaussian.** If $f(x) = (1/\sqrt{2\pi}\sigma)\exp(-x^2/2\sigma^2)$, $\tilde{f}(k) = \exp(-\sigma^2k^2/2)$.
**Multiple Choice: Be sure to put answers in boxes provided.**
(Sorry: no partial credit!)

**M1. (25 pts) Molecular Motors: Which Free Energy?**

An RNA polymerase molecular motor attached to a glass slide is pulling along a DNA molecule (transcribing it into RNA). The opposite end of the DNA molecule is attached to a bead which is being pulled by an optical trap with a constant external force $F$. Let the distance from the motor to the bead be $x$: thus the motor is trying to move to decrease $x$ and the force is trying to increase $x$.

This is a complicated example! It is under two constant forces ($F$ and pressure), and involves complicated chemistry and biology. Nonetheless, you know some things based on fundamental principles. Let us consider the optical trap and the distant fluid as being part of the external environment, and define the “system” as the local region of DNA, the RNA, motor, and the fluid and local molecules in a region immediately enclosing the region, as shown above. Without knowing anything further about the chemistry or biology in the system, which two of the following must be true on average, in all cases, according to basic laws of thermodynamics?

(T) (F) The total entropy of the universe (the system, bead, trap, laser beam . . .) must increase with time.

(T) (F) The entropy $S_s$ of the system must increase with time.

(T) (F) The total energy $E_T$ of the universe must decrease with time.

(T) (F) The energy $E_s$ of the system must decrease with time.

(T) (F) $G_s - Fx = E_s - TS_s + PV_s - Fx$ must decrease with time, where $G_s$ is the Gibbs free energy of the system.

Note: $F$ is a force, not the Helmholtz free energy. Precisely two of the answers are correct.

**Related formulæ:** $G = E - TS + PV$. 

2
M2. (30 pts) Wave Frequencies on a String.

At left is the sound power from a microphone measuring sound from a vibrating string. Assume the emitted power at a given frequency is proportional to the energy density in that mode.

(A) (10 points) \( \eta(x, t) \) is clearly a superposition of two standing waves, of amplitudes \( A_1 \) and \( A_3 \) for frequencies \( \omega_1 \) and \( \omega_3 \). What is the ratio of the amplitudes \( |A_3/A_1| \), given that the energy density in the two channels is the same (from the figure above)?

(a) \( |A_3/A_1| = 1 \).
(b) \( |A_3/A_1| = 1/3 \).
(c) \( |A_3/A_1| = 3 \).
(d) \( |A_3/A_1| = 1/9 \).
(e) \( |A_3/A_1| = 9 \).

Answer

Related formulae: Standing wave \( \eta(x, t) = A \sin(kx) \cos(\omega t) \) (for example).

\[
U = \int_0^L (1/2)\tau(\partial\eta/\partial x)^2 + (1/2)\lambda_0(\partial\eta/\partial t)^2 \, dx; \quad \text{standing} = (1/4)\tau k^2 A^2 L = (1/4)\lambda_0^2 \omega^2 A^2 L.
\]
The figures below show candidate initial conditions $\eta(x, 0)$ (solid) and velocities $\partial\eta/\partial t (x, t)$ (dashed) for waves on strings of length $L$.

(B) (20 points) There are a variety of ways to get the power spectrum shown in part (A). Cross out the two figures below which are NOT possible initial conditions, because they either will evolve with the wrong frequencies, or are not compatible with energy-conserving boundary conditions (fixed, free, or mixed). [Checking the amplitudes will take a lot of time: you can rule out the wrong ones using wavelengths and nodal positions alone.]

Note: precisely two should be crossed out. If you want to check amplitudes, you'll need to know the fundamental frequency $\omega_1$ in each case is 0.5 radians per second.

**Related formulae:** Standing wave $k_n = n\pi/L$ (fixed, free); $k_n = (2n - 1)\pi/2L$ (mixed). $\omega = ck = \sqrt{\tau/\lambda_0} k$. 
M3. (30 pts) P-V Diagrams.

A monatomic ideal gas in a piston is cycled around the path in the P-V diagram shown. Leg a cools at constant volume by connecting to a heat bath at $T_c$; leg b heats at constant pressure by connecting to a heat bath at $T_h$; leg c compresses at constant temperature while remaining connected to the bath at $T_h$. Which of the following are true? (5 points each.)

(T) (F) The cycle is reversible: no net entropy is created in the universe.

(T) (F) The cycle acts as a refrigerator, using work from the piston to draw energy from the cold bath into the hot bath, cooling the cold bath.

(T) (F) The cycle acts as an engine, transferring heat from the hot bath to the cold bath and doing positive net work on the outside world.

(T) (F) The work done per cycle has magnitude $|W| = P_0 V_0 |4 \log 4 - 3|$.

(T) (F) The heat transferred into the cold bath, $Q_c$ has magnitude $|Q_c| = (9/2) P_0 V_0$.

(T) (F) The heat transferred from the hot bath $Q_h$, plus the net work $W$ done by the piston onto the gas, equals the heat $Q_c$ transferred into the cold bath.

**Related formulæ:** $PV = N k_B T$, $U = (3/2) N k_B T$, $\Delta S = Q / T$, $W = - \int P dV$, $\Delta U = Q + W$. Notice that the signs of the various terms depend on convention (heat flow out vs. heat flow in): you should figure the signs on physical grounds.
Short Answer: Show Your Work

S1. (30 pts) Photon diffusion in the Sun.
Most of the fusion energy generated by the Sun is produced near its center. The Sun is $7 \times 10^5$ km in radius. Convection probably dominates heat transport in approximately the outer third of the Sun, but it is believed that energy is transported through the inner portions (say to a radius $R = 5 \times 10^8$ m) through a random walk of X-ray photons. (A photon is a quantized package of energy: you may view it as a particle which always moves at the speed of light $c$. Ignore for this problem the index of refraction of the Sun.) Assume that the mean free path $\ell$ for the photon is $\ell = 5 \times 10^{-5}$ m. About how many random steps $N$ will the photon take of length $\ell$ to get to the radius $R$ where convection becomes important? About how many years $\Delta t$ will it take for the photon to get there? (You may assume for this problem that the photon takes steps in random directions, each of equal length given by the mean-free path.)

\[ N = \text{_______________} \]

\[ \Delta t = \text{_______________} \text{ years} \]

Related formulae: $c = 3 \times 10^8$ m/s; $\langle x^2 \rangle \sim 2Dt$; $\langle s_n^2 \rangle = n\sigma^2 = n\langle s_1^2 \rangle$.
There are $31,556,925.9747 \sim \pi \times 10^7 \sim 3 \times 10^7$ seconds in a year.
S2. (30 pts) Two–state system.

Consider the statistical mechanics of a tiny object with only two discrete states:∗ one of energy $E_1$ and the other of higher energy $E_2 > E_1$.

(S2.A) (15 pts) Boltzmann probability ratio. Find the ratio of the equilibrium probabilities $\rho_2/\rho_1$ to find our system in the two states, when weakly coupled to a heat bath of temperature $T$. What is the limiting probability as $T \to \infty$? As $T \to 0$?

\[
\frac{\rho_2(T)}{\rho_1(T)} = \frac{\rho_2(0)}{\rho_1(0)} = \frac{\rho_2(\infty)}{\rho_1(\infty)} = \\
\]

Related formulæ: Boltzmann probability = $Z(T) \exp(-E/kT) \propto \exp(-E/kT)$.

∗ Visualize this as a tiny biased coin, which can be in the ‘heads’ or ‘tails’ state but has no other internal vibrations or center of mass degrees of freedom. Many systems are well described by large numbers of these two–state systems: some paramagnets, carbon monoxide on surfaces, glasses at low temperatures, …
(S2.B) (15 pts) **Probabilities and averages.** Use the normalization of the probability distribution (the system must be in one or the other state) to find $\rho_1$ and $\rho_2$ separately. (That is, solve for $Z(T)$ in the ‘related formula’ for part (A).) What is the average value of the energy $E$?

$$\rho_1(T) = \underline{\text{________________________}}$$

$$\rho_2(T) = \underline{\text{________________________}}$$

$$\langle E \rangle = \underline{\text{____________________________________}}$$
S3. (20 pts) Elastic Deformation Tensors.

A cube is deformed in the $xy$ plane according to the mapping $(x, y, z) \rightarrow (x + y, y, z)$.

(S3.A) (10 pts) Elastic tensor. Find the displacement field $u(x, y, z)$ and the elastic strain tensor $\varepsilon$ (including the geometric nonlinearity).

\[
\begin{align*}
\mathbf{u}(x, y, z) &= \begin{pmatrix}
\end{pmatrix}, \\
\varepsilon &= \begin{pmatrix}
\end{pmatrix}, \\
\text{Related formulæ: } \varepsilon_{ij} &= \frac{1}{2} (\partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k).
\end{align*}
\]
The view of the undistorted cube from the top (along the $z$ axis) is shown at left. We showed on the homework that if you include the geometric nonlinearity, rotating a body gave zero strain. More generally, it’s true that rotating an already deformed body doesn’t change the strain tensor.

(S3.B) (10 pts) **Pictures.** The pictures below are views from the top of a variety of cubes: the cubes are only deformed and rotated in the $xy$ plane. Circle the two pictures that have the strain tensor requested above.
S4. (35 pts) Solving Diffusion: Fourier and Green.

An initial density profile \( \rho(x, t = 0) \) is perturbed slightly away from a uniform density \( \rho_0 \), as shown at left. The density obeys the diffusion equation \( \frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2} \), where \( D = 0.001 \text{ m}^2/\text{s} \). The lump centered at \( x = 5 \) is a Gaussian \( \exp(-x^2/2)/\sqrt{2\pi} \), and the wiggle centered at \( x = 15 \) is a smooth envelope function multiplying \( \cos(10x) \).

(S4.A) (10 pts) **Fourier.** As a first step in guessing how the pictured density will evolve, let’s consider just a cosine wave. If the initial wave were \( \rho_{\cos}(x, 0) = \cos(10x) \), what would it be at \( t = 10 \text{s} \)?

\[
\rho_{\cos}(x, 10) = \ldots
\]

**Related formulæ:** \( \tilde{\rho}(k, t) = \tilde{\rho}(k, t') \tilde{G}(k, t - t'); \tilde{G}(k, t) = \exp(-Dk^2t) \).
(S4.B) (15 pts) **Green.**
As a second step, let’s check how long it would take to spread out as far as the Gaussian on the left. If the wave at some earlier time $-t_0$ were a $\delta$ function at $x = 0$, $\rho(x, -t_0) = \delta(x)$, what choice of the time elapsed $t_0$ would yield a Gaussian $\rho(x, 0) = \exp(-x^2/2)/\sqrt{2\pi}$ for the given diffusion constant $D = 0.001m^2/s$?

$t_0 = \underline{\text{ }}$.

**Related formulae:** $\rho(x, t) = \int \rho(y, t')G(y - x, t - t') dy$;
$G(x, t) = (1/\sqrt{4\pi Dt}) \exp(-x^2/(4Dt))$.  

(S4.C) (10 pts) **Pictures.** Now consider time evolution for the next ten seconds. The initial density profile $\rho(x,t=0)$ is again shown at left. Which of the following represents the density at $t = 10s$? (Hint: compare $t = 10s$ to the time $t_0$ from part (B).)

- **(A)**
- **(B)**
- **(C)**
- **(D)**
- **(E)**

**Answer**

Related formulæ: $\langle x^2 \rangle \sim 2Dt$;