Random Walks

- Molecules diffusing through the air undergo a random walk: straight lines between collisions.

- Density of Molecules = (Probability) (Number)

- Derive
  \[ \frac{\partial \rho}{\partial t} = D \nabla^2 \rho \]
  from random walks?

Free Body Diagram
**Coin Flips**

- $N$ coin flips
- $X_i = +1$ heads
- $X_i = -1$ tails
- $S_N = \sum_{i=1}^{N} X_i$

**How far does it get after $N$ steps?**

$$\langle S_N \rangle = 0 = \sum_{i=1}^{N} \langle X_i \rangle$$

Positive & Negative Cancel

$$\sqrt{\langle S_N^2 \rangle}$$ Better measure of typical distance

**Induction**

$$\langle S_N^2 \rangle = \langle (S_{N-1} + X_N)^2 \rangle = \langle S_{N-1}^2 \rangle + 2\langle S_{N-1} X_N \rangle + \langle X_N^2 \rangle$$

$$= \langle S_{N-1}^2 \rangle + 2(\langle S_{N-1} \rangle(1) + \langle S_{N-1} \rangle(-1)) + 1$$

$$= \langle S_{N-1}^2 \rangle + 1 = \ldots = N$$

Average root-mean-square distance moved $= \sqrt{N}$

Just like diffusion equation? “RMS”

Probability distribution

$$p(s, N) = \frac{1}{2} p(s+1, N-1) + \frac{1}{2} p(s-1, N-1)$$

Will return later! Link to $\frac{\partial p}{\partial t} = D \nabla^2 p$.
Drunkard's Walk

- Starts at lamppost $\vec{S}_0 = \vec{0}$
- Each step $\vec{X}_N$ random direction, length $L$

$$\vec{S}_N = \vec{S}_{N-1} + \vec{X}_N$$

$$\langle \vec{S}_N \rangle = 0$$

$$\langle \vec{S}_N^2 \rangle = \langle (\vec{S}_{N-1} + \vec{X}_N)^2 \rangle$$

$$= \langle (\vec{S}_{N-1})^2 \rangle + 2 \langle \vec{S}_{N-1} \cdot \vec{X}_N \rangle + \langle \vec{X}_N^2 \rangle$$

= $NL^2$

RMS Distance $\sim \sqrt{N}L$
General Case

Let $S_N$ be the sum of random variables $X_i$. Let each $X_i$ have the same probability distribution $\xi(x)$, with zero mean and RMS = $a$:

$$\int \xi(x) = 1, \quad \int x \xi(x) = 0,$$

$$\int x^2 \xi(x) = a^2$$

Then $\langle S_N \rangle = 0$ and

$$\langle S_N^2 \rangle = \langle (S_{N-1} + X_N)^2 \rangle$$

$$= \langle S_{N-1}^2 \rangle + 2 \langle S_{N-1} \rangle \langle X_N \rangle + \langle X_N^2 \rangle$$

$$= \langle S_{N-1}^2 \rangle + a^2 = Na^2$$

and the evolution law for $p$ is

$$p(s,N) = \int p(s',N-1) \xi(s-s') ds'$$

How likely at $s'$ before
How likely last step went from $s'$ to $s$
The Continuum Limit & the Diffusion Equation

- Suppose the time between steps is $\Delta t$.
  \[ s(t+\Delta t) = s(t) + x(t) \]
  \[ p(s, t+\Delta t) = \int p(s', t) \xi(s-s') \, ds' \]
  \[ = \int p(s-z, t) \xi(z) \, dz \]

- Suppose that the step size is very small compared to the sizes of variation in $p$.
  \[ \Rightarrow \text{Taylor expand in } z \]

\[ p(s, t+\Delta t) = \int [p(s, t) + z \frac{\partial p}{\partial s} + \frac{z^2}{2} \frac{\partial^2 p}{\partial s^2}] \xi(z) \, dz \]
\[ = p(s, t) \int \xi(z) \, dz + \frac{\partial p}{\partial s} \int z \xi(z) \, dz + \frac{\partial^2 p}{\partial s^2} \int \frac{z^2}{2} \xi(z) \, dz \]
So, \[ \frac{\partial P}{\partial t} \Delta t = \frac{a^2}{2} \frac{\partial^2 P}{\partial x^2} \]

\[ \Rightarrow \quad \frac{\partial P}{\partial t} = \frac{a^2}{2 \Delta t} \frac{\partial^2 P}{\partial x^2} \]

"Microscopic" Derivation of Diffusion Equation

Consequences:

1. \( D = \frac{a^2}{2 \Delta t} \). Can estimate diffusion constant from collision lengths and times

2. Random walks, sums of random variables, become Gaussian in continuum limit

Central Limit Theorem

\[ \rho_n(x) \rightarrow \frac{1}{\sqrt{2\pi} \sigma} e^{-x^2/2\sigma^2} \]

Gaussian also called "Normal" distribution

as \( N \rightarrow \infty \), with \( \sigma^2 = Na^2 \)

["Almost perfect" for \( N \approx 5 \)]