Fourier Analysis

Math Questions for Today!

1. Can we write a general solution \( y(x,t) \) in terms of standing waves?
   - Fourier Series

2. Why are sines, cosines, and exponentials so often the solutions in homogeneous problems?
   - [Group Representations in hiding]

We won't derive the answers, but we will rephrase the questions in FUNCTION SPACE.

Consider the space \( S \) of all initial conditions \( y(x,0) \), defined on \( \mathbb{R}^1 \).

In many ways, \( S \) is an infinite-dimensional analogy to \( \mathbb{R}^3 \), the space of three-dimensional vectors. Motivation for many otherwise miraculous formulas...
<table>
<thead>
<tr>
<th>( \mathbb{R}^3 )</th>
<th>( \mathbb{S} )</th>
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</thead>
<tbody>
<tr>
<td><strong>Vector Space of Positions</strong></td>
<td><strong>Space of Functions</strong></td>
</tr>
<tr>
<td>Domain</td>
<td>( \hat{\mathbf{r}} = (r_1, r_2, r_3) ) one real per ( x \in {1, 2, 3} )</td>
</tr>
<tr>
<td>Dot Product</td>
<td>( \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = r_1 s_1 + r_2 s_2 + r_3 s_3 )</td>
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<tr>
<td>Distance</td>
<td>(</td>
</tr>
<tr>
<td>Basis</td>
<td>( \hat{x}_1, \hat{x}_2, \hat{x}_3 )</td>
</tr>
<tr>
<td>Norm 1</td>
<td>( \hat{x}_i \cdot \hat{x}_i = 1 )</td>
</tr>
<tr>
<td>Orthogonal</td>
<td>( \hat{x}_i \cdot \hat{x}_j = 0 ) if ( i \neq j )</td>
</tr>
<tr>
<td>Coefficients</td>
<td>( r_n = \hat{\mathbf{r}} \cdot \hat{x}_n )</td>
</tr>
<tr>
<td>Completeness</td>
<td>( \hat{\mathbf{r}} = \sum r_n \hat{x}_n )</td>
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<tr>
<td></td>
<td>( \eta_n = \frac{\hat{f}_n \cdot \eta}{\hat{f}_n \cdot \hat{f}_n} = \int_0^L \eta(x) \sqrt{\frac{L}{2}} \sin(k_n x) dx )</td>
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<td></td>
<td>( a_n = \sqrt{\frac{L}{2}} \eta_n = \int_0^L \eta(x) \frac{L}{2} \sin(k_n x) dx )</td>
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<td></td>
<td>( \eta = \sum \eta_n \hat{f}_n = \sum \frac{L}{2} \eta_n \sin(k_n x) )</td>
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<td>( = \sum a_n \sin(k_n x) )</td>
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*Much math to prove it*

Any initial condition (on the right functions space) can be expanded...
\[ \hat{r}_1 x_1 + \hat{r}_2 x_2 \text{ closest point to } \hat{r} \text{ in } (x_1, x_2) \text{ plane} \]

\[ \sum_{n=1}^{N} \eta_n \hat{F}_n \text{ minimizes distance } \int_0^L (\eta(x) - \sum_{n=1}^{N} \eta_n F_n)^2 \, dx \]

E&H 1.7.2, PS3

What does this imply about the solutions to the wave equation, \( \eta(x, t) \)?

**Exercise:** Remember that \( \sin(k_n x) \cos(\omega_n t) \) satisfies the wave equation for \( \omega_n = c k_n = c n \pi / L \).

If \( \eta(x, 0) = \sum a_n \sin(k_n x) \) starts at rest, what is \( \eta(x, t) \)?

\[ \eta(x, t) = \sum a_n \sin(k_n x) \cos(\omega_n t) \]

\[ t = 0 \quad \eta(x, 0) \leftarrow \]

\[ \frac{\partial \eta}{\partial t} = \sum -a_n \omega_n \sin(k_n x) \sin(\omega_n t) = 0 \text{ at } t = 0 \]

**Exercise:** If a relaxed string \( \eta(x, 0) = 0 \) is given an impulse at \( t = 0 \), \( \eta(x, 0) = \sum \nu_n \sin(k_n x) \), what is \( \eta(x, t) \)?

\[ \eta(x, t) = \sum \frac{\nu_n}{\omega_n} \sin(k_n x) \sin(\omega_n t) \]

**Answer to 0:**

By superposition, setting \( b_n = \frac{\nu_n}{\omega_n} \), the general solution is

\[ \eta(x, t) = \sum_n \sin(k_n x) \left( a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \right) \]
Question 2:

Why are the solutions cosines and sines?

Because $e^{i\omega t} = \cos \omega t + i \sin \omega t$

Why is $e^{i\omega t}$ special?

It's the eigenfunction of translations.

Translational Symmetries:
if $\eta(x,t)$ is a solution

- Time Independent
- Homogeneous

- $T_\Delta \eta$ and $R_\Delta \eta$ shift functions to the right in time and space, by amount $\Delta$

- Equations of motion respect these symmetries

- $T_\Delta$ and $R_\Delta$ are linear mappings from $\mathcal{S}$ to $\mathcal{S}$

Analogy to $\mathbb{R}^3$: linear mappings are 3x3 matrices $\vec{\mathbf{F}} \mapsto \mathbf{M} \cdot \vec{\mathbf{F}} = \sum M_{ij} \vec{\mathbf{F}}_j$

Eigenvectors of $\mathbf{M}$ are often useful $\mathbf{M} \cdot \vec{\mathbf{e}}_n = \lambda_n \vec{\mathbf{e}}_n$
What are the eigenfunctions of $T_\Delta$?

$$T_\Delta(\xi(t)) = \xi'(t) = \lambda \xi(t)$$

$$\xi(t) = e^{i\omega t}$$

$$\xi(t-\Delta) = e^{i\omega(t-\Delta)} = e^{-i\omega \Delta} \xi(t)$$

$$\lambda = e^{-i\omega \Delta}$$

Can we use these to prove that there are solutions of the form $\xi_\omega(x,t) = e^{i\omega t} g(x)$? Real, imaginary parts give $\cos(\omega t), \sin(\omega t)$ solutions. Start with any solution $\xi(x,t)$.

$\xi(x,t-\Delta) = T_\Delta(\xi)$ is also a solution for all $\Delta$.

Superimpose these solutions, dividing by $\lambda$

$$\xi_\omega(x,t) = \sum_{-\infty}^{\infty} \xi(x,t-\Delta) / e^{-i\omega \Delta} d\Delta$$

$$= \sum_{-\infty}^{\infty} e^{i\omega \Delta} \xi(x,t-\Delta) d\Delta$$

$$= \sum_{-\infty}^{\infty} e^{i\omega(t-\Delta)} g(x) [-d\Delta]$$

Complex version of Fourier coefficient:

$$= e^{i\omega t} \left[ -\sum_{-\infty}^{\infty} e^{-i\omega \xi} \xi(x,\xi) d\xi \right] g(x)$$

Similarly, for an infinite string, can prove that there are solutions $e^{ikx}$.