How to Find New Laws

String Wave Equation

1. Pick an Order Parameter Field
   - Represents those local variables of importance and interest
   - For us: \( \eta(x, t) \), vertical height of string

2. Imagine the most general possible law
   \[ F(\eta, x, t, \frac{\partial \eta}{\partial x}, \frac{\partial^2 \eta}{\partial x^2}, \frac{\partial^3 \eta}{\partial x^3}, \ldots, \frac{\partial^N \eta}{\partial x^N}, \ldots) = 0 \]
   general, nonlinear function \( F \)

3. Restrict attention to long lengths and times
   \[ \text{Slow variations} \]
   \[ \Rightarrow D \text{ big} \]
   \[ \Rightarrow \frac{\partial^n \eta}{\partial x^n} \sim \frac{1}{D^N} \text{ small} \]

We'll ignore derivatives cubic and higher

Exercise

What is this function?
[Get the 2π's right!]
What is its 400\(^{th}\) derivative \( \frac{d^{400} \eta}{2x^{400}} \)?
Symmetries of Waves on a String

Assume an infinite, stretched string.

Exercise: What symmetries does this system have?

If \( y(x,t) \) is a solution:
- Reflection along \( x \)
- Time reversal
- Reflection along \( y \)

Discrete
- \( y(-x,t) \)
- \( y(x,-t) \)
- \( -y(x,t) \)

Continuous
- Homogeneous (Translation along \( x \))
- Time independent (Translate time)
- Sideways motion (Translation along \( y \))

\( y(x+\Delta,t) \)
\( y(x,t+\Delta) \)
\( y(x,t)+\Delta \)
Apply continuous symmetries

Equations of motion $F=0$ must have the same continuous symmetries as the system.

Sideways Motion

$\Delta \eta(x,t) = \eta + \Delta$

$F(\eta, x, t, \frac{\partial \eta}{\partial x}, \frac{\partial^2 \eta}{\partial x^2}, ...)$

$F_{\text{independent of } \eta}$

Homogeneous

$\Delta \eta(x,t) = \eta(x-\Delta, t)$

$F_{\text{of } \eta \text{ at } x+\Delta = F_{\text{ of } \eta \text{ at } x}}$

$F(\eta(x+\Delta), x+\Delta, t, \frac{\partial \eta}{\partial x}(x+\Delta), ...)$

$F(\eta(x), x, \frac{\partial \eta}{\partial x}, ...)$

$F_{\text{independent of } x}$

Time Independent

$F_{\text{independent of } t}$
(5) (often) Assume order parameter small

For us, ignore terms quadratic & higher in \( \eta \)

\[
A + B \frac{\partial \eta}{\partial x} + C \frac{\partial^2 \eta}{\partial x^2} + D \frac{\partial^3 \eta}{\partial x^3} + E \frac{\partial^4 \eta}{\partial x^4} + F \frac{\partial^5 \eta}{\partial x^5} = 0
\]

(6) Apply discrete symmetries

\( \eta \) must either stay the same or change sign under discrete symmetries

Reflection Along \( y \) \( \tilde{\eta} = -\eta \Rightarrow \tilde{\eta} \rightarrow -\tilde{\eta} \)
The terms change sign but constant \( A \)

\( \Rightarrow A = 0 \)

Reflection Along \( x \) \( \tilde{\eta}(x,t) = \eta(-x,t) \)

Terms odd in \( \frac{\partial}{\partial x} \) change sign
Terms even stay unchanged

Choose \( \tilde{\eta} \rightarrow \tilde{\eta} \) \( \Rightarrow B, F = 0 \)

Time Reversal Invariant: \( \tilde{\eta}(x,t) = \eta(x,-t) \Rightarrow C = 0 \)

\[
D \frac{\partial^2 \eta}{\partial x^2} + E \frac{\partial^2 \eta}{\partial t^2} = 0 \quad \Rightarrow \frac{\partial^2 \eta}{\partial t^2} = -\frac{D}{E} \frac{\partial^2 \eta}{\partial x^2} \quad \text{If } D/E < 0
\]

Wave Equation!