Solving the Wave Equation Numerically

Consider a string of length $L$ that is shaken up and down at the left end $\eta(0,t) = f(t)$ and is fixed in position $\eta(L,t) \equiv 0$ at the right end.

$$\frac{\partial^2 \eta}{\partial t^2} = c^2 \frac{\partial^2 \eta}{\partial x^2} \quad (1)$$

To solve this equation numerically, we must discretize the string into chunks of size $\delta x$ in space, and take small, discrete time steps $\delta t$ in time.

1. Derive the approximate formula for the second derivative

$$\frac{\partial^2 \eta}{\partial x^2} \approx \frac{\eta(x + \delta x) - 2\eta(x) + \eta(x - \delta x)}{\delta x^2} \quad (2)$$

from the approximate formula for the first derivative

$$\frac{\partial \eta}{\partial x}(x_0) \approx \frac{\eta(x_0 + \epsilon/2) - \eta(x_0 - \epsilon/2)}{\epsilon} \quad (3)$$

(Hint: pick $\epsilon = \delta x$ and $x_0 = x \pm \delta x/2$. It may help to draw a picture of where you are evaluating the first and second derivatives.)

2. Applying this approximate formula to the wave equation (1), show that we can write the future position of the string in terms of the past and present. If our wire is broken up into $N$ chunks of size $\delta x = N/L$,

$$x_0 \equiv 0, \quad x_1 = \delta x, \quad \ldots \quad x_N = N\delta x \equiv L \quad (4)$$

show that

$$\eta(x_i, t + \delta t) \approx 2\eta(x_i, t) - \eta(x_i, t - \delta t) + (c\delta t/\delta x)^2 (\eta(x_{i+1}) - 2\eta(x_i) + \eta(x_{i-1})) \quad (5)$$

Notice that this equation applies for $i = 1 \ldots N-1$, but not for $i = 0$ or $i = N$. These boundary conditions have to be supplied separately: in our case, fixed on the right, forced on the left.
3. Write a program (using Matlab, Mathematica, a spreadsheet, or any other method of your choice) to solve this wave equation with \( L = 10\text{m} \), \( c = 2\text{m/s} \), \( \delta x = 0.1\text{m} \), \( \delta t = 0.025\text{s} \), and

\[
 f(t) = \exp(-(4 - 2t)^2/2) .
\]  

(6)

Use the evolution equation (5) and the initial conditions

\[
 \eta(x_i, 0) \equiv \eta(x_i, -\delta t) \equiv 0 .
\]  

(6)

(For debugging, you might want to start with \( \delta x = 0.5\text{m} \) and \( \delta t = 0.125\text{s} \).) When should the pulse center hit the right end of the string? Plot the pulse shape when the center is partway to the wall, when your analysis says it should be hitting the wall, and after it is reflected. Where do you think the energy is stored when the pulse is at the wall?)