Fourier Lab Instructions
Physics 218: Waves and Thermodynamics
Fall 2001, James P. Sethna

This lab has several sections. The later sections may be declared optional by your instructor. Also, there are several analytic problems throughout the lab: your instructor may wish you to do these problems before you come to lab.

In this lab we will use a Fast Fourier Transform (FFT). This is a clever numerical algorithm. It starts with N equally spaced data points over an interval of length $N\delta y$, so $u_n = u(y_n)$ with $y_n = n\delta y$. It returns the complex Fourier transform as a set of complex numbers $v_m = v(z_m)$ at $z_m = 2\pi m/(N\delta y)$, for $m = -N/2+1$ to $N/2$ *:

$$v(z) = \sum_n u_n \exp(-izy_n). \tag{FL1}$$

For the first parts of our lab, we'll be going from real space to k-space, finding the Fourier transform of functions $f(x)$, which we'll write $\tilde{f}(k)$. For convenience, we'll arrange our points to be roughly symmetric about the origin, so our interval runs from $-L/2$ to $L/2$, our points start at $-L/2+\delta x$, and end at $L/2$, with $\delta x = L/N$. We'll mostly be working with $N = 100$ points, so the FFT will give us

$$\tilde{f}^{FFT}(k) = \sum_{n=-49}^{50} f(n\delta x)\exp(-ikn\delta x). \tag{FL2}$$

The formula for the Fourier series is similar, but not quite the same:

$$\tilde{f}^{analytic}(k) = (1/L)\int_{-L/2}^{L/2} f(x)\exp(-ikx)dx. \tag{FL3}$$

Analytic Problem #1: By changing the sum (FL2) to an integral, show that

$$\tilde{f}^{analytic}(k) \approx 1/N \tilde{f}^{FFT}(k). \tag{FL4}$$

(The sum in (FL2) weights each term with weight one; it should weight each term by the width of the interval between points δx. The formula (FL3) has a $1/L$ outside.)

Cosine Wave with 32 points: aliasing.

Obviously the computer must work with finite numbers of points, which is why we must discretize our function into N points $x_n = n\delta x$. Obviously the computer has to return the Fourier tranform again at a finite number of points k_m. The first question we'll address is why the points $k_m = 2\pi m/L$ with $m = -N/2+1$ to $N/2$ are the natural choices for points to return.

Run Wave32, using the instructions on the other handout. You'll see a cosine wave $\cos(kx+\delta)$, evaluated at 32 points from $x = -10$ to 10 as described above. What's different

* Presuming N is even.

about waves with one of the special values $k_m = 2\pi m/L$ from intermediate values of k? (Try varying δ.) The FFT is particularly appropriate for investigating periodic functions of period L: other functions will have blurred Fourier peaks as we'll see later.

You'll note that the slider only allows values of k in a certain range. Verify that the maximum and minima of this range corresponds to $m = +-N/2$, roughly the range of k values that will be returned by the FFT. By looking at the cosine wave at the maximum and the minimum values, explain why the FFT doesn't calculate $m = -N/2$, but starts at $m = -N/2 + 1$.

Try typing in $k = 10$, outside of this range. Why does it look the way it does? Calculate the wavelength for $k = 10$, and the distance between plotted points. Find a value of k inside the range that has the same plot as $k = 10$.

Analytic Problem #2: Show in general that $\cos((k+K)n\delta x) = \cos(kn\delta x)$ for $K = 2\pi/\delta x$. (This should follow from the periodicity of cosine.) This is called *aliasing*.

Calculate K for our problem with $N = 32$ and $L = 20$. What is the wavevector $10 - K$? Does $\cos((10-K)x)$ look the same as $\cos(10x)$ did?

Fourier Transform of Cosine Wave.

To go from the Fourier coefficients $\tilde{f}^{analytic}(k_m)$ back to real space, we use an *inverse* Fourier transform:

$$f(x) = \sum_m \tilde{f}^{analytic}(k_m)\exp(ik_m x). \tag{FL5}$$

Notice that there is a minus sign difference between this formula and the forward Fourier transforms (FL1), (FL2), and (FL3).

Analytic Problem #3: Writing $\cos(k_m x)$ as a sum of two exponentials, guess what the FFT will give as a series $\tilde{f}(k_n)$ for cosine. [Hint: the formula (FL5) is a sum over N exponentials: can you set $N-2$ of the coefficients to zero and get your formula for cosine? From Problem #1, how are the FFT fourier series coefficients related to those of the analytic Fourier series (FL5)?] What should the FFT for $\sin(kx)$ be?

Run FourierWave. On the left you should see a cosine wave $f(x) = \cos(kx + \delta)$ evaluated at 100 points; on the right you should see the FFT output $\tilde{f}(k)$ (black is the real part, red is the imaginary part). Does it agree with your expectation? Check the height and the position. Change δ until you form $\sin(kx)$. Were your guesses correct?

Change k to various values k_m: notice the nice clean Fourier peaks. Functions which are not periodic have their Fourier spectra smeared by these sidebands.

Change k to the maximum value $K/2$ discussed above. What does the FFT do?

Fourier Transform of Gaussian Packet.

Run FourierPacket. On the left you see a Gaussian packet in real space

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-(x-x_0)^2/2\sigma^2\right). \tag{FL6}$$

(The prefactor $\frac{1}{\sqrt{2\pi}\sigma}$ just makes the integral equal to one.) Gaussians are one of the most useful functions in physics. On the right you see its Fourier coefficients $\tilde{f}(k)$. How does the width of the Fourier transform change as the packet width σ is changed?

Note that the Fourier transform of a Gaussian looks like a Gaussian.

Analytic Problem #4: Show that the Fourier transform of a Gaussian is a Gaussian. Start with the analytical definition

$$\tilde{f}(k) \sim \int_{-\infty}^{\infty} \exp(-x^2/2\sigma^2) \exp(-ikx) dx. \tag{FL7}$$

Combine the two exponentials, complete the square so that the integrand looks like $\exp\left(-a(x-b)^2 - ab^2\right)$, and pull out the term looking like $\exp -ab^2$ which doesn't depend on x. Then change variables to $y = x + ik\sigma^2$.* Without doing the integral over y, you should now be able to see that the answer is some constant times a Gaussian in k.

Verify that the width of the Gaussian in k in your formula is the inverse of the width of the original Gaussian (FL7) in x. Verify this using the FFT as well.

Analytic Problem #5: Show, for a general shape $f(x)$, that the Fourier transform of $f(x - \Delta)$,

$$\int_{-\infty}^{\infty} f(x - \Delta) \exp(-ikx) dx. \tag{FL8}$$

is $\exp(-ik\Delta)\tilde{f}(k)$. [Hint: change variables to $y = x - \Delta$.]

For a small value of σ (so $\tilde{f}(k)$ is wide enough to see well) change x_0. Do the real and imaginary parts of the FFT of the packet look like $\cos(kx_0)$ and $-\sin(kx_0)$ times the FFT of the packet at $x_0 = 0$?

Fourier Transform of White Noise.

White light is a mixture of light of all frequencies. White noise is a mixture of sound of all frequencies, with constant average power per unit frequency. The hissing noise you hear on radio and TV between stations is approximately white noise: there are a lot more high frequencies than low ones, so it sounds high-pitched.

What kind of time signal would generate white noise? Run FourierNoise. On the left you see a series of random numbers: each $f_n = f(n\delta x)$ is chosen independently as a random number between minus one and one. Zoom in (unlike the other graphs, this one isn't an animation so the buttons on xmgr work), and see the curve jumping around at random. (Hitting the button to the right of the magnifying glass will restore the original scale.) On the right you see the power spectrum. The average power at a certain frequency in a signal is proportional to $|\tilde{f}(k)|^2$, which is what we plot on the right side. Notice that the power is noisy, but on average is independent of frequency. White noise is usually due to random, uncorrelated fluctuations in time.

What can we learn about Fourier transforms using this example? Try zooming into the Fourier graph. It should also look like random numbers, unless you zoom in near $k = 0$. What happens there?

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* This is really a cheat. The integral now goes from $-\infty + ik\sigma^2$ to $+\infty + ik\sigma^2$. You need to know some complex analysis to know that you can deform the contour in the complex plane, using Cauchy's theorem. Physicists often get away with this kind of cheating...

Analytic Problem #6: Prove that the power spectrum is symmetric when $k \to -k$. You can do this in three steps. (1) Prove that the real part of $\tilde{f}$ is symmetric, from equation (FL3). [Hint: what is the real part of $\exp(-ikx)$?] (2) Prove that the imaginary part of $\tilde{f}$ is antisymmetric. (3) Use the fact that if $z = u + iv$ then $|z|^2 = u^2 + v^2$ to prove that the power spectrum is symmetric.

Fourier Transform Twice.

We wrote equation (FL1) in terms of $u(y)$ and $v(z)$ to emphasize that you can use FFT's both to go from real space to Fourier space and back again. The only difference is the minus sign missing in (FL5). If you run FourierSquared, you'll see on the left a somewhat arbitrary function (generated by smoothing between random numbers), and on the right the Fourier transform of the Fourier transform of the function (FL1 with $y = x$ and $z = k$, from real space to k-space, followed by FL1 with $y = k$ and $z = x$ from k-space back to real space). Notice that the final function is flipped $x \to -x$ (with a bit of a jump at the end because I was sloppy with the function).

Analytic Problem #7: Show that the Fourier transform of the Fourier transform, $\tilde{\tilde{f}}(x)$ gives back the function $f(x)$ inverted $x \to -x$. Roughly speaking, this comes from the fact that the formula for the inverse transform (FL5) is about the same as that for the forward transform (FL1), except for the minus sign in the exponential is missing in (FL5). One simple method to show this explicitly is to work outward in both directions from $\tilde{f}$. Use equation (FL5) to write a formula for $f(-x)$. Compare that formula to the formula you would get from (FL1) for $\tilde{\tilde{f}}(x)$ with $y = k$, $u = \tilde{f}$, $v = \tilde{\tilde{f}}$, and $z = x$.