Physics 218—Exam II (November 7) Fall 2001

Formula Sheet

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Sound Waves in Three Dimensions.

$\rho\partial^2\mathbf{u}/\partial t^2 = -\nabla p$, $p = -B\nabla\cdot\mathbf{u}$, $\partial^2 p/\partial t^2 = c^2\nabla^2 p$ with $c = \sqrt{B/\rho}$, $\partial^2\mathbf{u}/\partial t^2 = c^2\nabla^2\mathbf{u}$.
Spherical waves: $p(\mathbf{r}, t) = f(|\mathbf{r}| - ct)/|\mathbf{r}|$.
Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where the index of refraction $n = \sqrt{\epsilon\mu}$ is c/v.
Phase shift π on reflection where the index of refraction increases (*e.g.*, light off glass).
Intensity along the direction of propagation $I = p\partial\mathbf{u}/\partial t$,
energy density $E = (\rho/2)(\partial u/\partial t)^2 + p^2/(2B)$.
If $p(t) = \sum_n \tilde{p}_n \exp(i\omega_n t)$ and $\rho(t) = \sum_m \tilde{\rho}_m \exp(i\omega_m t)$, then the total power is the sum of the power in each frequency channel: $\sum_n (-i\omega/2)\tilde{p}_n\tilde{\rho}_n^*$.

Interference and Diffraction.

Double Slit. Phase difference $\phi = 2\pi d\sin(\theta)/\lambda = kd\sin(\theta)$.
Intensity $I_{av} = 4I_0\cos^2(\phi/2) = 4I_0\cos^2(kd\sin\theta/2)$ (I_0 single slit intensity).
Constructive for $d\sin\theta = 0, \pm\lambda, \pm 2\lambda, \ldots$, destructive for $d\sin\theta = \pm\lambda/2, \pm 3\lambda/2, \ldots$.
Multiple slits. $I_{av} = I_0\sin^2(N\phi/2)/\sin^2(\phi/2)$; principle maxima at $\phi = 0, 2\pi, 4\pi$, destructive at $\phi = 2m\pi/N$ with m any integer *except* $0, \pm N, \pm 2N, \ldots$.
Diffraction. If the slit opening is $f(x)$, $I_{av} \propto |\tilde{f}(k\sin\theta)|^2$.
The Fourier transform of a shifted function $f(x - \Delta)$ is $\exp(-i\Delta k)\tilde{f}(k)$.
Single wide slit. $I_{av} = I_{center}\sin^2\alpha/\alpha^2$ with $\alpha = ak\sin(\theta)/2$.

Tensor Notation.

Einstein convention: $a_{ijk\ell}b_{imno} = \sum_{i=1}^{3} a_{ijk\ell}b_{imno}$.
Dot product $\mathbf{a}\cdot\mathbf{b} = a_i b_i$, matrix applied to vector $(M\mathbf{x})_i = M_{ij}x_j$, matrix multiplication $(MN)_{ij} = M_{ik}N_{kj}$, trace $Tr(M) = M_{ii}$.
Laplacian $\nabla^2 f = \partial_i\partial_i f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$, divergence $\nabla\cdot\mathbf{v} = \partial_i v_i$.
Identity tensor δ_{ij}, equals one if $i = j$, zero otherwise.
Totally antisymmetric tensor $\epsilon_{ijk} : \epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kij}$. $\epsilon_{123} = 1 = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{321} = 1 = \epsilon_{213} = \epsilon_{132} = -1$, zero if any index repeats.
$(\mathbf{a}\times\mathbf{b})_i = \epsilon_{ijk}a_j b_k$, $(\nabla\times\mathbf{v})_i = \epsilon_{ijk}\partial_j v_k$, $\det M = \epsilon_{ijk}\epsilon_{\ell mn}M_{i\ell}M_{jm}M_{kn}$.
$\delta_{ii} = 3$, $\epsilon_{ijk}\delta_{jk} = 0$, $\epsilon_{ijk}\epsilon_{ijk} = 6$, $\epsilon_{ijk}\epsilon_{ij\ell} = 2\delta_{k\ell}$, $\epsilon_{ijm}\epsilon_{k\ell m} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}$.

Elasticity Theory.

Stress tensor $\sigma_{ij}\hat{\mathbf{n}}_j$ = Force/Area across surface perpendicular to $\hat{\mathbf{n}}$. $\sigma_{ij} = \sigma_{ji}$ because torques on small volumes must vanish. Force on a small volume $F_i = \partial_j\sigma_{ij}$. For hydrostatic pressure P, $\sigma_{ij} = -P\delta_{ij}$.
Strain tensor $\varepsilon_{ij} = (1/2)\left(\partial_i u_j + \partial_j u_i + \partial_i u_k\partial_j u_k\right)$, where the last term (the geometric nonlinearity) is usually ignored. $\varepsilon_{ij} = \varepsilon_{ji}$. The strain tensor for uniform stretching $-\Delta V/V = 3\Delta L/L$ would be $\varepsilon_{ij} = (\Delta L/L)\delta_{ij}$.

Tensor of elasticity $c_{ijk\ell}$ gives Hooke's law for anisotropic media, $\sigma_{ij} = c_{ijk\ell}\varepsilon_{k\ell}$. $c_{ijk\ell} = c_{jik\ell} = c_{ij\ell k} = c_{k\ell ij}$. There are 21 possible independent elastic constants.
The elastic energy density $E = (1/2)\sigma_{ij}\varepsilon_{ij} = (1/2)c_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell}$.
Isotropic moduli. The bulk modulus K is the same as B for fluids: $P = -K(\Delta V/V)$.
Under a shear by an angle θ, $E = (1/2)\mu\theta^2$.
Under unconstrained stretching, $F = Y\Delta L/L$, and $\Delta W/W = -\sigma\Delta L/L$, where here σ is Poisson's ratio and *not* the strain. $K = 2\mu/3 + \lambda$, $\sigma = \lambda/2(\mu+\lambda)$, and $Y = (2\mu^2 + 3\lambda\mu)/(\mu+\lambda)$.
Isotropic Tensors. $c_{ijk\ell} = \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + \lambda\delta_{ij}\delta_{k\ell}$. $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$. $E = \mu\varepsilon_{ij}\varepsilon_{ij} + (\lambda/2)(\varepsilon_{kk})^2$.
Wave equations. $\rho_0\partial^2 u_i/\partial t^2 = \partial_j\sigma_{ij} = (1/2)c_{ijk\ell}\partial_j(\partial_k u_\ell + \partial_\ell u_k)$. For isotropic media, $\rho_0\partial^2 u_i/\partial t^2 = (\lambda+\mu)\partial_i\partial_j u_j + \mu\partial_j\partial_j u_i$, or $\rho_0\partial^2\mathbf{u}/\partial t^2 = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u}$.
Decomposing $\mathbf{u} = \mathbf{u}_T + \mathbf{u}_L$ with $\nabla\cdot\mathbf{u}_T = 0$ and $\nabla\times\mathbf{u}_L = 0$, we have $\partial^2\mathbf{u}_L/\partial t^2 = c_L^2\nabla^2\mathbf{u}_L$ and $\partial^2\mathbf{u}_T/\partial t^2 = c_T^2\nabla^2\mathbf{u}_T$, with $c_T = \sqrt{\mu/\rho_0}$ and $c_L = \sqrt{(\lambda+2\mu)/\rho}$.

Electromagnetic Waves.

Maxwell's Equations.

$$\nabla\cdot\mathbf{D} = 4\pi\rho$$
$$\nabla\times\mathbf{H} - \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{J}$$
$$\nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0$$
$$\nabla\cdot\mathbf{B} = 0$$

with $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. ϵ and μ can be tensors for anisotropic media.
Plane waves. Linearly polarized about $\mathbf{E}_0 = (0, E_y, E_z)$:

$$\mathbf{E}(\mathbf{r},t) = E_y^0\hat{\mathbf{y}}e^{i(kx-\omega t)} + E_z^0\hat{\mathbf{z}}e^{i(kx-\omega t)}$$
$$\mathbf{B}(\mathbf{r},t) = -E_z^0\hat{\mathbf{y}}e^{i(kx-\omega t)} + E_y^0\hat{\mathbf{z}}e^{i(kx-\omega t)}$$

Circularly polarized wave:

$$\mathbf{E}(\mathbf{r},t) = E^0\hat{\mathbf{y}}e^{i(kx-\omega t)} + iE^0\hat{\mathbf{z}}e^{i(kx-\omega t)}$$
$$\mathbf{B}(\mathbf{r},t) = -iE^0\hat{\mathbf{y}}e^{i(kx-\omega t)} + E^0\hat{\mathbf{z}}e^{i(kx-\omega t)}$$

Formulas from Prelim I.

Trigonometry $f = \omega/2\pi$, and $k = 2\pi/\lambda$. $\exp(iz) = \cos(z) + i\sin(z)$, $\cos(z) = (\exp(iz) + \exp(-iz))/2$, and $\sin(z) = (\exp(iz) - \exp(-iz))/(2i)$.
Wave Equation Solutions. The wave equation

$$\partial^2\eta/\partial t^2 = c^2\partial^2\eta/\partial x^2$$

has a traveling wave solution $\eta(x,t) = f(x \pm ct)$, a standing-wave solution $\eta(x,t) = A\sin(kx)\sin(\omega t)$, and (as a special case) a traveling sine wave $\eta(x,t) = A\exp(i(kx-\omega t))$, where $\omega/k = c$.
Fourier Transform of a Gaussian. If $f(x) = (1/\sqrt{2\pi}\sigma)\exp\left(-x^2/2\sigma^2\right)$, $\tilde{f}(k) = \exp\left(-\sigma^2k^2/2\right)$.