NAME: ________________________________

Multiple Choice (40 pts): ____ x 10pts = __________

Short Answer: 

S1 (25 pts) ________________

S2 (35 pts) ________________

TOTAL ________________
Two loudspeakers a distance 6m apart emit spherical sound waves, in phase, at frequency $\omega$. The speed of sound in air is $v = 340 \text{ m/s}$. An experimentalist measures the net sound at a point 8m directly in front of one of the loudspeakers. What is the lowest frequency $\omega_D$ at which she measures destructive interference from the two speakers? What is the ratio $I_{av}/I_8$ of the intensity of the sound she measures at that frequency, compared to the sound she measures when the more distant speaker is shut off?

(A) $\omega_D = 85\pi$, $I_{av}/I_8 = 1/5$.  
(B) $\omega_D = 85\pi$, $I_{av}/I_8 = 1/4$.  
(C) $\omega_D = 170\pi$, $I_{av}/I_8 = 9/400$.  
(D) $\omega_D = 170\pi$, $I_{av}/I_8 = 1/25$.  
(E) $\omega_D = 340\pi$, $I_{av}/I_8 = 16/25$.  

Answer __________________________________________________________________________________________
M2. (10 pts) Tensor Notation.

Suppose \( \mathbf{B} = \nabla \times \mathbf{A} \). Which of the following are correct formulas for \( \mathbf{B}^2 \)? (For example, the energy contained in a magnetic field is \( \mathbf{B}^2/8\pi \).)

(A) \( \varepsilon_{ijk} \partial_j A_k \varepsilon_{i\ell m} \partial_\ell A_m \).

(B) \( (\delta_{j\ell} \delta_{km} - \delta_{jm} \delta_{k\ell}) (\partial_j A_k)(\partial_\ell A_m) \).

(C) \( (\partial_j A_k)^2 - (\partial_j A_k \partial_k A_j) \).

(D) All of the above.

(E) None of the above.

Answer

DID YOU LOOK AT ALL THE ANSWERS?
M3. (10 pts) Interference

A coherent laser beam impinges on a slit of width $a$. An intensity pattern is viewed on a distant screen: the center has intensity $I_0$ and the peak width (distance between the nearest minima) is $\Delta Y$. The slit is broadened to $2a$. What is the new intensity $I_{\text{doubled}}$ and peak minimum separation $\Delta Y'$? You may assume that the angles are small, so $\sin \theta \approx \theta$.

(A) $I' = 4I_0$, $\Delta Y' = \Delta Y/2$.
(B) $I' = 2I_0$, $\Delta Y' = \Delta Y/2$.
(C) $I' = 2I_0$, $\Delta Y' = \Delta Y/4$.
(D) $I' = 4I_0$, $\Delta Y' = 2\Delta Y$.
(E) $I' = 2I_0$, $\Delta Y' = 2\Delta Y$.

Answer:
M4. (10 pts) Elastic travelling wave.

An isotropic elastic medium with density $\rho$ and moduli $\lambda$ and $\mu$ fills the half space $x > 0$. The boundary of this medium is wiggled with displacement field

$$u(0, y, z) = (f(t), g(t), h(t)),$$

generating an elastic wave travelling to the right (positive $x$ direction). What is the displacement $u(x, y, z, t)$ for $x > 0$?

(A) $u(x, y, z, t) = (0, g(t - x/c), h(g - x/c))$.

(B) $u(x, y, z, t) = (f(t - x/\sqrt{(\lambda + 2\mu)/\rho}), g(t - x/\sqrt{\mu/\rho}), h(t - x/\sqrt{\mu/\rho}))$.

(C) $u(x, y, z, t) = (f(t - x/\sqrt{\mu/\rho}), g(t - x/\sqrt{(\lambda + 2\mu)/\rho}), h(t - x/\sqrt{(\lambda + 2\mu)/\rho}))$.

(D) $u(x, y, z, t) = (f(x - \sqrt{\mu/\rho} t), g(x - \sqrt{(\lambda + 2\mu)/\rho} t), h(x - \sqrt{(\lambda + 2\mu)/\rho} t))$.

(E) $u(x, y, z, t) = (f(t - x/\sqrt{\mu/\rho}), g(t - y/\sqrt{(\lambda + 2\mu)/\rho}), h(t - z/\sqrt{(\lambda + 2\mu)/\rho}))$.

Answer
A sound wave generator generates a triangular pressure air wave moving toward the right down a hollow tube, as shown in the figure above. The triangles repeat forever with wavelength $L$. The maximum displacement of the wave is $A$, the velocity of sound is $v$, and the bulk modulus for air is $B$.

(S1.A) (7 pts) What is the intensity (energy per unit time per unit area) traveling down the tube? (The relevant formulas are on the formula sheet.)
The figure shows the Fourier series for our wave truncated at $n = \pm 2$ and $n = \pm 4$.

(S1.B) (8pts) We now want to decompose this intensity into different frequencies. What would the time average $I_{av}^n$ for the intensity of a travelling plane wave of wave vector $k_n$ and amplitude $a_n$, $u_n(x, t) = a_n \sin (k_n(x - vt))$? (Leave your answer in terms of $a_n$ and $k_n$.)
The Fourier series for the displacement of the wave is

\[ u(x) = \sum_{n=0}^{\infty} a_n \sin(k_n(x - vt)) \]

with \( k_n = \frac{2\pi n}{L} \). The Fourier coefficients are \( a_n = 0 \) for \( n \) even, and

\[ a_n = \frac{(-1)^{(n-1)/2} A}{\pi^2 n^2} \]

for \( n \) odd.

(S1.C) (10 pts) Verify explicitly that the sum of the intensities per frequency channel \( n \) you calculated in part (B) equals the total intensity you calculated in part (A). This is the theorem you proved in the crumpling paper problem using the orthogonality of different Fourier modes.
S2. (35 pts) Waves on a Thin Wire.

A plane wave of wave vector $k$ passes along the $\hat{x}$ direction through a thin sheet of width $\Delta Y = W$. The wire width $W$ is thin compared to the wavelength, so $kW << 1$. The wave at $t = 0$ is approximately given by

$$u(x, y, z) = (A \sin(kx), -\sigma Aky \cos(kx), 0).$$

(S2.A) (15 pts) Compute the strain tensor $\varepsilon(x, y, z, t)$ for this displacement field, ignoring the geometric nonlinearity. Write it out as a $3 \times 3$ matrix.
(S2.C) (20 pts) The wire is isotropic, with elastic moduli $\lambda$ and $\mu$. Write the stress tensor for the wire as a $3 \times 3$ matrix. (Warning: the $z$ components may not be zero.)
Sound Waves in Three Dimensions.
\[ \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = -\nabla p, \quad p = -B \nabla \cdot \mathbf{u}, \quad \frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p \text{ with } c = \sqrt{B/\rho}, \quad \frac{\partial^2 \mathbf{u}}{\partial t^2} = c^2 \nabla^2 \mathbf{u}. \]
Spherical waves: \( p(r, t) = f(|r| - ct)/|r| \).

Snell’s law: \( n_1 \sin \theta_1 = n_2 \sin \theta_2 \), where the index of refraction \( n = \sqrt{\varepsilon \mu} \) is \( c/v \).

Intensity along the direction of propagation \( I = \rho \partial \mathbf{u}/\partial t \),
evapor density \( E = (\rho/2)(\partial \mathbf{u}/\partial t)^2 + p^2/(2B) \).

If \( p(t) = \sum n \tilde{p}_n \exp(i \omega_n t) \) and \( \rho(t) = \sum m \tilde{\rho}_m \exp(i \omega_m t) \), then the total power is the sum of the power in each frequency channel: \( \sum n (-i \omega/2) \tilde{p}_n \tilde{p}_n^{\ast} \).

Interference and Diffraction.

Double Slit. Phase difference \( \phi = 2\pi d \sin(\theta)/\lambda = kd \sin(\theta) \).

Intensity \( I_{av} = 4I_0 \cos^2(\phi/2) = 4I_0 \cos^2(kd \sin \theta/2) \) (I0 single slit intensity).

Constructive for \( d \sin \theta = 0, \pm \lambda, \pm 2\lambda, \ldots \), destructive for \( d \sin \theta = \pm \lambda/2, \pm 3\lambda/2, \ldots \).

Multiple slits. \( I_{av} = I_0 \sin^2(N \phi/2)/\sin^2(\phi/2) \); principle maxima at \( \phi = 0, 2\pi, 4\pi \), destructive at \( \phi = 2m\pi/N \) with \( m \) any integer except \( 0, \pm N, \pm 2N, \ldots \).

Diffraction. If the slit opening is \( f(x) \), \( I_{av} \propto |\tilde{f}(k \sin \theta)|^2 \).

The Fourier transform of a shifted function \( f(x - \Delta) \) is \( \exp(-i \Delta k) \tilde{f}(k) \).

Single wide slit. \( I_{av} = I_{center} \sin^2 \alpha/\alpha^2 \) with \( \alpha = ak \sin(\theta)/2 \).

Tensor Notation.

Einstein convention: \( a_{ijkl} b_{imno} = \sum_{i=1}^{3} a_{ijkl} b_{imno} \).

Dot product \( \mathbf{a} \cdot \mathbf{b} = a_i b_i \), matrix applied to vector \( (M \mathbf{x})_i = M_{ij} x_j \), matrix multiplication \( (MN)_{ij} = M_{ik} N_{kj} \), trace \( \text{Tr}(M) = M_{ii} \).

Laplacian \( \nabla^2 f = \partial_i \partial_i f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f \), divergence \( \nabla \cdot \mathbf{v} = \partial_i v_i \).

Identity tensor \( \delta_{ij} \), equals one if \( i = j \), zero otherwise.

Totally antisymmetric tensor \( \epsilon_{ijk} : \epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{kij} \). \( \epsilon_{123} = 1 = \epsilon_{231} = \epsilon_{312} = 1, \epsilon_{321} = 1 = \epsilon_{213} = \epsilon_{132} = -1 \), zero if any index repeats.

\( (\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k \), \( (\nabla \times \mathbf{v})_i = \epsilon_{ijk} \partial_j v_k \), det \( M = \epsilon_{ijk} \epsilon_{lmn} M_{ij} M_{lm} M_{kn} \).

\( \delta_{ii} = 3 \), \( \epsilon_{ijk} \delta_{jk} = 0 \), \( \epsilon_{ijk} \epsilon_{ijk} = 6 \), \( \epsilon_{ijk} \epsilon_{jkl} = 2 \delta_{kl} \), \( \epsilon_{ijm} \epsilon_{kml} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \).

Elasticity Theory.

Stress tensor \( \sigma_{ij} \mathbf{n}_j = \text{Force/Area across surface perpendicular to } \mathbf{n} \). \( \sigma_{ij} = \sigma_{ji} \) because torques on small volumes must vanish. Force on a small volume \( F_i = \partial_j \sigma_{ij} \). For hydrostatic pressure \( P \), \( \sigma_{ij} = -P \delta_{ij} \).

Strain tensor \( \varepsilon_{ij} = (1/2) (\partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k) \), where the last term (the geometric nonlinearity) is usually ignored. \( \varepsilon_{ij} = \varepsilon_{ji} \). The strain tensor for uniform stretching \( -\Delta V/V = 3\Delta L/L \) would be \( \varepsilon_{ij} = (\Delta L/L) \delta_{ij} \).
Tensor of elasticity $c_{ijkl}$ gives Hooke’s law for anisotropic media, $\sigma_{ij} = c_{ijkl} \varepsilon_{kl}$. $c_{ijkl} = c_{iilk} = c_{kijl}$. There are 21 possible independent elastic constants.

The elastic energy density $E = (1/2)\sigma_{ij}\varepsilon_{ij} = (1/2)c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$.

Isotropic moduli. The bulk modulus $K$ is the same as $B$ for fluids: $P = -K(\Delta V/V)$.

Under unconstrained stretching, $F = Y\Delta L/L$, and $\Delta W/W = -\sigma\Delta L/L$, where here $\sigma$ is Poisson’s ratio and not the strain. $K = 2\mu/3 + \lambda$, $\sigma = \lambda/2(\mu + \lambda)$, and $Y = (2\mu^2 + 3\lambda\mu)/(\mu + \lambda)$.

Isotropic Tensors. $c_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}$. $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$. $E = \mu\varepsilon_{ij}\varepsilon_{ij} + (\lambda/2)(\varepsilon_{kk})^2$.

Wave equations. $\rho_0 \partial^2 u_i/\partial t^2 = \partial_j\sigma_{ij} = (1/2)c_{ijkl}\partial_j(\partial_k u_l + \partial_l u_k)$. For isotropic media, $\rho_0 \partial^2 u_i/\partial t^2 = (\lambda + \mu)\partial_i u_j + \mu\partial_j u_i$, or $\rho_0 \partial^2 \mathbf{u}/\partial t^2 = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}$.

Decomposing $\mathbf{u} = \mathbf{u}_T + \mathbf{u}_L$ with $\nabla \cdot \mathbf{u}_T = 0$ and $\nabla \times \mathbf{u}_L = 0$, we have $\partial^2 \mathbf{u}_L/\partial t^2 = c_L^2 \nabla^2 \mathbf{u}_L$ and $\partial^2 \mathbf{u}_T/\partial t^2 = c_T^2 \nabla^2 \mathbf{u}_T$, with $c_T = \sqrt{\mu/\rho_0}$ and $c_L = \sqrt{(\lambda + 2\mu)/\rho}$.

Electromagnetic Waves.

Maxwell’s Equations.
\[
\nabla \cdot \mathbf{D} = 4\pi \rho \\
\nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c} \mathbf{J} \\
\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \\
\n\nabla \cdot \mathbf{B} = 0
\]

with $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$. $\varepsilon$ and $\mu$ can be tensors for anisotropic media.

Plane waves. Linearly polarized about $\mathbf{E}_0 = (0, E_y, E_z)$:
\[
\mathbf{E}(r, t) = E_0^0 \hat{y} e^{i(kx - \omega t)} + E_0^z \hat{z} e^{i(kx - \omega t)} \\
\mathbf{B}(r, t) = -E_0^z \hat{y} e^{i(kx - \omega t)} + E_0^0 \hat{z} e^{i(kx - \omega t)}
\]

Circularly polarized wave:
\[
\mathbf{E}(r, t) = E_0^0 \hat{y} e^{i(kx - \omega t)} + i E_0^z \hat{z} e^{i(kx - \omega t)} \\
\mathbf{B}(r, t) = -i E_0^z \hat{y} e^{i(kx - \omega t)} + E_0^0 \hat{z} e^{i(kx - \omega t)}
\]

Formulas from Prelim I.

Trigonometry $f = \omega/2\pi$, and $k = 2\pi/\lambda$. $\exp(iz) = \cos(z) + i \sin(z)$, $\cos(z) = (\exp(iz) + \exp(-iz))/2$, and $\sin(z) = (\exp(iz) - \exp(-iz))/(2i)$.

Wave Equation Solutions. The wave equation
\[
\partial^2 \eta/\partial t^2 = c^2 \partial^2 \eta/\partial x^2
\]
has a traveling wave solution $\eta(x, t) = f(x \pm ct)$, a standing-wave solution $\eta(x, t) = A \sin(kx) \sin(\omega t)$, and (as a special case) a traveling sine wave $\eta(x, t) = A \exp(i(kx - \omega t))$, where $\omega/k = c$.

Fourier Transform of a Gaussian. If $f(x) = (1/\sqrt{2\pi}\sigma)^2 \exp(-x^2/2\sigma^2)$, $\tilde{f}(k) = \exp(-\sigma^2 k^2/2)$.