Physics 218—Exam II (November 7) **Fall 2001**
James P. Sethna

NAME:______________________________

Multiple Choice (40 pts):_____ **x 10pts =** _______________

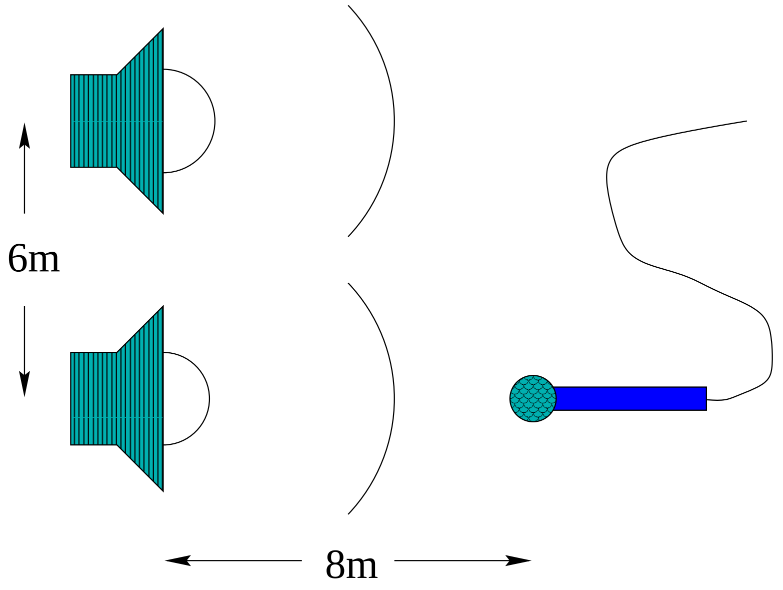
Short Answer: **S1 (25 pts)** ___________________

S2 (35 pts) ___________________

TOTAL ___________________

Multiple Choice: Be sure to put answers in boxes provided. (Sorry: no partial credit!)

M1. (10 pts) Interference of Spherical Waves.



Two loudspeakers a distance 6m apart emit spherical sound waves, in phase, at frequency ω. The speed of sound in air is $v = 340$ m/s. An experimentalist measures the net sound at a point $8m$ directly in front of one of the loudspeakers. What is the lowest frequency ω_D at which she measures destructive interference from the two speakers? What is the ratio I_{av}/I_8 of the intensity of the sound she measures at that frequency, compared to the sound she measures when the more distant speaker is shut off?

(A) $\omega_D = 85\pi$, $I_{av}/I_8 = 1/5$.

(B) $\omega_D = 85\pi$, $I_{av}/I_8 = 1/4$.

(C) $\omega_D = 170\pi$, $I_{av}/I_8 = 9/400$.

(D) $\omega_D = 170\pi$, $I_{av}/I_8 = 1/25$.

(E) $\omega_D = 340\pi$, $I_{av}/I_8 = 16/25$.

Answer

M2. (10 pts) Tensor Notation.

Suppose $\mathbf{B} = \nabla \times \mathbf{A}$. Which of the following are correct formulas for $\mathbf{B}^2$? (For example, the energy contained in a magnetic field is $\mathbf{B}^2/8\pi$.)

(A) $\varepsilon_{ijk}\partial_j A_k \varepsilon_{i\ell m}\partial_\ell A_m$.

(B) $(\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell})(\partial_j A_k)(\partial_\ell A_m)$.

(C) $(\partial_j A_k)^2 - (\partial_j A_k \partial_k A_j)$.

(D) All of the above.

(E) None of the above.

Answer

DID YOU LOOK AT ALL THE ANSWERS?

M3. (10 pts) Interference

A coherent laser beam impinges on a slit of width a. An intensity pattern is viewed on a distant screen: the center has intensity I_0 and the peak width (distance between the nearest minima) is ΔY. The slit is broadened to $2a$. What is the new intensity $I_{doubled}$ and peak minimum separation $\Delta Y'$? You may assume that the angles are small, so $\sin\theta \approx \theta$.

(A) $I' = 4I_0$, $\Delta Y' = \Delta Y/2$.
(B) $I' = 2I_0$, $\Delta Y' = \Delta Y/2$.
(C) $I' = 2I_0$, $\Delta Y' = \Delta Y/4$.
(D) $I' = 4I_0$, $\Delta Y' = 2\Delta Y$.
(E) $I' = 2I_0$, $\Delta Y' = 2\Delta Y$.

Answer

M4. (10 pts) Elastic travelling wave.

An isotropic elastic medium with density ρ and moduli λ and μ fills the half space $x > 0$. The boundary of this medium is wiggled with displacement field

$$\mathbf{u}(0, y, z) = (f(t), g(t), h(t))\,,$$

generating an elastic wave travelling to the right (positive x direction).
What is the displacement $\mathbf{u}(x, y, z, t)$ for $x > 0$?

(A) $\mathbf{u}(x, y, z, t) = (0, g(t - x/c), h(g - x/c))$.
(B) $\mathbf{u}(x, y, z, t) = (f(t - x/\sqrt{(\lambda + 2\mu)/\rho}), g(t - x/\sqrt{\mu/\rho}), h(t - x/\sqrt{\mu/\rho}))$.
(C) $\mathbf{u}(x, y, z, t) = (f(t - x/\sqrt{\mu/\rho}), g(t - x/\sqrt{(\lambda + 2\mu)/\rho}), h(t - x/\sqrt{(\lambda + 2\mu)/\rho}))$.
(D) $\mathbf{u}(x, y, z, t) = (f(x - \sqrt{\mu/\rho}\,t), g(x - \sqrt{(\lambda + 2\mu)/\rho}\,t), h(x - \sqrt{(\lambda + 2\mu)/\rho}\,t))$.
(E) $\mathbf{u}(x, y, z, t) = (f(t - x/\sqrt{\mu/\rho}), g(t - y/\sqrt{(\lambda + 2\mu)/\rho}), h(t - z/\sqrt{(\lambda + 2\mu)/\rho}))$.

Answer

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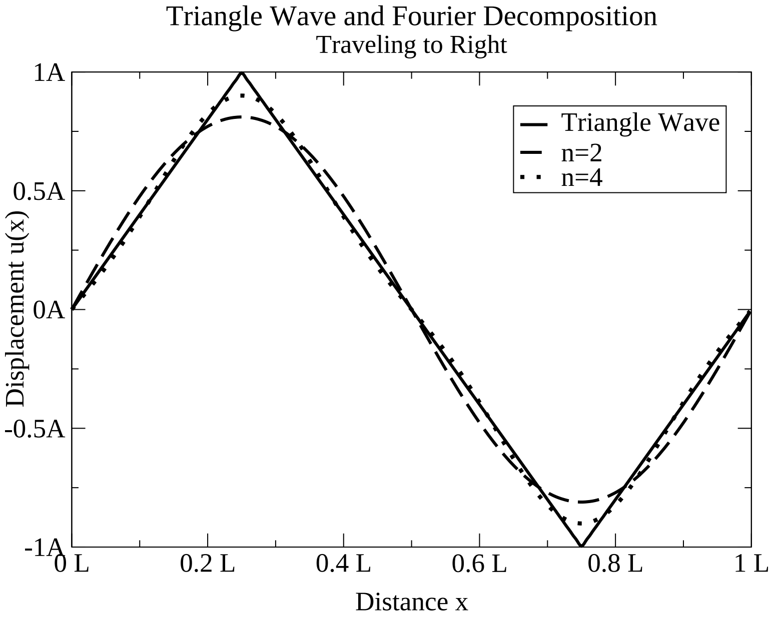
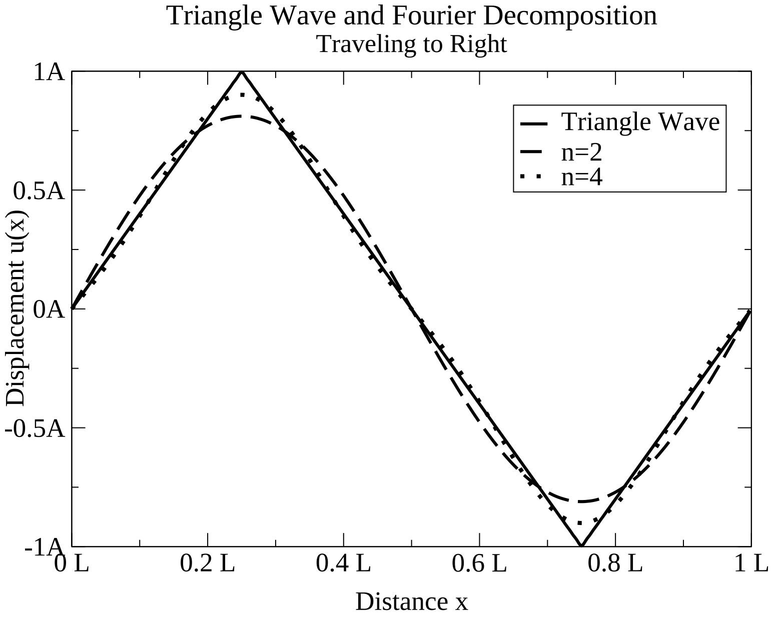
Short Answer: Show Your Work

S1. (25 pts) Sawtooth Wave.



A sound wave generator generates a triangular pressure air wave moving toward the right down a hollow tube, as shown in the figure above. The triangles repeat forever with wavelength L. The maximum displacement of the wave is A, the velocity of sound is v, and the bulk modulus for air is B.

(S1.A) (7 pts) What is the intensity (energy per unit time per unit area) traveling down the tube? (The relevant formulas are on the formula sheet.)



The figure shows the Fourier series for our wave truncated at $n = \pm 2$ and $n = \pm 4$.

(S1.B) (8pts) We now want to decompose this intensity into different frequencies. What would the time average I_n^{av} for the intensity of a travelling plane wave of wave vector k_n and amplitude a_n, $u_n(x,t) = a_n \sin(k_n(x - vt))$? (Leave your answer in terms of a_n and k_n.)

The Fourier series for the displacement of the wave is

$$u(x) = \sum_{n=0}^{\infty} a_n \sin(k_n(x - vt))$$

with $k_n = 2\pi n/L$. The Fourier coefficients are $a_n = 0$ for n even, and

$$a_n = (-1)^{(n-1)/2} 8A/(\pi^2 n^2)$$

for n odd.

(S1.C) (10 pts) Verify explicitly that the sum of the intensities per frequency channel n you calculated in part (B) equals the total intensity you calculated in part (A).[*] You'll need the formula $\pi^2/8 = 1 + 1/3^2 + 1/5^2 + 1/7^2 + \ldots$

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[*] This is the theorem you proved in the crumpling paper problem using the orthogonality of different Fourier modes.

S2. (35 pts) Waves on a Thin Wire.

A plane wave of wave vector k passes along the $\hat{x}$ direction through a thin sheet of width $\Delta Y = W$. The wire width W is thin compared to the wavelength, so $kW << 1$. The wave at $t = 0$ is approximately given by

$$\mathbf{u}(x, y, z) = (A \sin(kx), -\sigma A k y \cos(kx), 0).$$

(S2.A) (15 pts) Compute the strain tensor $\varepsilon(x, y, z, t)$ for this displacement field, ignoring the geometric nonlinearity. Write it out as a 3×3 matrix.

(S2.C) (20 pts) The wire is isotropic, with elastic moduli λ and μ. Write the stress tensor for the wire as a 3×3 matrix. (Warning: the z components may not be zero.)

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Formula Sheet

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Sound Waves in Three Dimensions.

$\rho \partial^2 \mathbf{u}/\partial t^2 = -\nabla p$, $p = -B\nabla \cdot \mathbf{u}$, $\partial^2 p/\partial t^2 = c^2 \nabla^2 p$ with $c = \sqrt{B/\rho}$, $\partial^2 \mathbf{u}/\partial t^2 = c^2 \nabla^2 \mathbf{u}$.
Spherical waves: $p(\mathbf{r}, t) = f(|\mathbf{r}| - ct)/|\mathbf{r}|$.
Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where the index of refraction $n = \sqrt{\epsilon\mu}$ is c/v.
Phase shift π on reflection where the index of refraction increases (*e.g.*, light off glass).
Intensity along the direction of propagation $I = p\partial \mathbf{u}/\partial t$,
energy density $E = (\rho/2)(\partial u/\partial t)^2 + p^2/(2B)$.
If $p(t) = \sum_n \tilde{p}_n \exp(i\omega_n t)$ and $\rho(t) = \sum_m \tilde{\rho}_m \exp(i\omega_m t)$, then the total power is the sum of the power in each frequency channel: $\sum_n (-i\omega/2)\tilde{p}_n \tilde{\rho}_n^*$.

Interference and Diffraction.

Double Slit. Phase difference $\phi = 2\pi d \sin(\theta)/\lambda = kd\sin(\theta)$.
Intensity $I_{av} = 4I_0 \cos^2(\phi/2) = 4I_0 \cos^2(kd\sin\theta/2)$ (I_0 single slit intensity).
Constructive for $d\sin\theta = 0, \pm\lambda, \pm 2\lambda, \ldots$, destructive for $d\sin\theta = \pm\lambda/2, \pm 3\lambda/2, \ldots$.
Multiple slits. $I_{av} = I_0 \sin^2(N\phi/2)/\sin^2(\phi/2)$; principle maxima at $\phi = 0, 2\pi, 4\pi$, destructive at $\phi = 2m\pi/N$ with m any integer *except* $0, \pm N, \pm 2N, \ldots$.
Diffraction. If the slit opening is $f(x)$, $I_{av} \propto |\tilde{f}(k\sin\theta)|^2$.
The Fourier transform of a shifted function $f(x - \Delta)$ is $\exp(-i\Delta k)\tilde{f}(k)$.
Single wide slit. $I_{av} = I_{center} \sin^2\alpha/\alpha^2$ with $\alpha = ak\sin(\theta)/2$.

Tensor Notation.

Einstein convention: $a_{ijk\ell} b_{imno} = \sum_{i=1}^{3} a_{ijk\ell} b_{imno}$.
Dot product $\mathbf{a} \cdot \mathbf{b} = a_i b_i$, matrix applied to vector $(M\mathbf{x})_i = M_{ij} x_j$, matrix multiplication $(MN)_{ij} = M_{ik} N_{kj}$, trace $Tr(M) = M_{ii}$.
Laplacian $\nabla^2 f = \partial_i \partial_i f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$, divergence $\nabla \cdot \mathbf{v} = \partial_i v_i$.
Identity tensor δ_{ij}, equals one if $i = j$, zero otherwise.
Totally antisymmetric tensor $\epsilon_{ijk} : \epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kij}$. $\epsilon_{123} = 1 = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{321} = 1 = \epsilon_{213} = \epsilon_{132} = -1$, zero if any index repeats.
$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$, $(\nabla \times \mathbf{v})_i = \epsilon_{ijk} \partial_j v_k$, $\det M = \epsilon_{ijk} \epsilon_{\ell mn} M_{i\ell} M_{jm} M_{kn}$.
$\delta_{ii} = 3$, $\epsilon_{ijk}\delta_{jk} = 0$, $\epsilon_{ijk}\epsilon_{ijk} = 6$, $\epsilon_{ijk}\epsilon_{ij\ell} = 2\delta_{k\ell}$, $\epsilon_{ijm}\epsilon_{k\ell m} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}$.

Elasticity Theory.

Stress tensor $\sigma_{ij} \hat{\mathbf{n}}_j$ = Force/Area across surface perpendicular to $\hat{\mathbf{n}}$. $\sigma_{ij} = \sigma_{ji}$ because torques on small volumes must vanish. Force on a small volume $F_i = \partial_j \sigma_{ij}$. For hydrostatic pressure P, $\sigma_{ij} = -P\delta_{ij}$.
Strain tensor $\varepsilon_{ij} = (1/2)\left(\partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k\right)$, where the last term (the geometric nonlinearity) is usually ignored. $\varepsilon_{ij} = \varepsilon_{ji}$. The strain tensor for uniform stretching $-\Delta V/V = 3\Delta L/L$ would be $\varepsilon_{ij} = (\Delta L/L)\delta_{ij}$.

Tensor of elasticity $c_{ijk\ell}$ gives Hooke's law for anisotropic media, $\sigma_{ij} = c_{ijk\ell}\varepsilon_{k\ell}$. $c_{ijk\ell} = c_{jik\ell} = c_{ij\ell k} = c_{k\ell ij}$. There are 21 possible independent elastic constants.
The elastic energy density $E = (1/2)\sigma_{ij}\varepsilon_{ij} = (1/2)c_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell}$.
Isotropic moduli. The bulk modulus K is the same as B for fluids: $P = -K(\Delta V/V)$.
Under a shear by an angle θ, $E = (1/2)\mu\theta^2$.
Under unconstrained stretching, $F = Y\Delta L/L$, and $\Delta W/W = -\sigma\Delta L/L$, where here σ is Poisson's ratio and *not* the strain. $K = 2\mu/3 + \lambda$, $\sigma = \lambda/2(\mu+\lambda)$, and $Y = (2\mu^2 + 3\lambda\mu)/(\mu+\lambda)$.
Isotropic Tensors. $c_{ijk\ell} = \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + \lambda\delta_{ij}\delta_{k\ell}$. $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$. $E = \mu\varepsilon_{ij}\varepsilon_{ij} + (\lambda/2)(\varepsilon_{kk})^2$.
Wave equations. $\rho_0\partial^2 u_i/\partial t^2 = \partial_j\sigma_{ij} = (1/2)c_{ijk\ell}\partial_j(\partial_k u_\ell + \partial_\ell u_k)$. For isotropic media, $\rho_0\partial^2 u_i/\partial t^2 = (\lambda+\mu)\partial_i\partial_j u_j + \mu\partial_j\partial_j u_i$, or $\rho_0\partial^2\mathbf{u}/\partial t^2 = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u}$.
Decomposing $\mathbf{u} = \mathbf{u}_T + \mathbf{u}_L$ with $\nabla\cdot\mathbf{u}_T = 0$ and $\nabla\times\mathbf{u}_L = 0$, we have $\partial^2\mathbf{u}_L/\partial t^2 = c_L^2\nabla^2\mathbf{u}_L$ and $\partial^2\mathbf{u}_T/\partial t^2 = c_T^2\nabla^2\mathbf{u}_T$, with $c_T = \sqrt{\mu/\rho_0}$ and $c_L = \sqrt{(\lambda+2\mu)/\rho}$.

Electromagnetic Waves.

Maxwell's Equations.

$$\begin{aligned}\nabla\cdot\mathbf{D} &= 4\pi\rho\\ \nabla\times\mathbf{H} - \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} &= \frac{4\pi}{c}\mathbf{J}\\ \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} &= 0\\ \nabla\cdot\mathbf{B} &= 0\end{aligned}$$

with $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. ϵ and μ can be tensors for anisotropic media.
Plane waves. Linearly polarized about $\mathbf{E}_0 = (0, E_y, E_z)$:

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) =& E_y^0\hat{\mathbf{y}}e^{i(kx-\omega t)} + E_z^0\hat{\mathbf{z}}e^{i(kx-\omega t)}\\ \mathbf{B}(\mathbf{r},t) =& -E_z^0\hat{\mathbf{y}}e^{i(kx-\omega t)} + E_y^0\hat{\mathbf{z}}e^{i(kx-\omega t)}\end{aligned}$$

Circularly polarized wave:

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) =& E^0\hat{\mathbf{y}}e^{i(kx-\omega t)} + iE^0\hat{\mathbf{z}}e^{i(kx-\omega t)}\\ \mathbf{B}(\mathbf{r},t) =& -iE^0\hat{\mathbf{y}}e^{i(kx-\omega t)} + E^0\hat{\mathbf{z}}e^{i(kx-\omega t)}\end{aligned}$$

Formulas from Prelim I.

Trigonometry $f = \omega/2\pi$, and $k = 2\pi/\lambda$. $\exp(iz) = \cos(z) + i\sin(z)$, $\cos(z) = (\exp(iz) + \exp(-iz))/2$, and $\sin(z) = (\exp(iz) - \exp(-iz))/(2i)$.
Wave Equation Solutions. The wave equation

$$\partial^2\eta/\partial t^2 = c^2\partial^2\eta/\partial x^2$$

has a traveling wave solution $\eta(x,t) = f(x \pm ct)$, a standing-wave solution $\eta(x,t) = A\sin(kx)\sin(\omega t)$, and (as a special case) a traveling sine wave $\eta(x,t) = A\exp(i(kx-\omega t))$, where $\omega/k = c$.
Fourier Transform of a Gaussian. If $f(x) = (1/\sqrt{2\pi}\sigma)\exp\left(-x^2/2\sigma^2\right)$, $\tilde{f}(k) = \exp\left(-\sigma^2k^2/2\right)$.