Physics 218—Exam I (October 5) Fall 2001
Formula Sheet

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Trigonometry. The angle addition formulas are $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and $\cos(A+B) = \cos A \cos B - \sin A \sin B$. For a sine wave, $f = \omega/2\pi$, and $k = 2\pi/\lambda$. The trig functions are related to complex exponentials: $\exp(iz) = \cos(z) + i\sin(z)$, $\cos(z) = (\exp(iz) + \exp(-iz))/2$, and $\sin(z) = (\exp(iz) - \exp(-iz))/(2i)$.

Orthonormality. $(2/\pi)\int_0^\pi \sin(m\theta)\sin(n\theta) = \delta_{mn}$, $(2/\pi)\int_0^\pi \cos(m\theta)\cos(n\theta) = \delta_{mn}$, $(2/\pi)\int_0^\pi \cos(m\theta)\sin(n\theta) = 0$, and $(1/2\pi)\int_0^{2\pi} \exp(im\theta)\exp(-in\theta) = \delta_{mn}$ where $\delta_{mn} = 0$ for $m \neq n$ and $\delta_{mn} = 1$ for $m = n$.

Fourier. The formula for the Fourier series coefficients $\tilde{f}(k_m)$ of a function $f(x)$ in an interval of length L is

$$\tilde{f}^{analytic}(k_m) = (1/L)\int_{-L/2}^{L/2} f(x)\exp(-ik_m x)dx.$$

where $k_m = 2\pi m/L$. The Fourier series can be summed to retrieve the original function, just as a vector can be reconstructed by multiplying its coefficients times the basis vectors:

$$f(x) = \sum_m \tilde{f}^{analytic}(k_m)\exp(ik_m x).$$

The Fourier transform is roughly the Fourier series as $L \to \infty$, apart from normalization. The Fourier transform of a Gaussian of width σ, $f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-x^2/2\sigma^2\right)$ is a Gaussian $\tilde{f}(k) = \exp\left(-\sigma^2 k^2/2\right)$ of width $1/\sigma$. The Fourier transform of a shifted function $f(x-\Delta)$ is $\exp(-i\Delta k)\tilde{f}(k)$.

Wave Equations. The wave equation

$$\partial^2\eta/\partial t^2 = c^2\partial^2\eta/\partial x^2$$

has a traveling wave solution $\eta(x,t) = f(x \pm ct)$, a standing-wave solution $\eta(x,t) = A\sin(kx)\sin(\omega t)$, and (as a special case) a traveling sine wave $\eta(x,t) = A\sin(kx - \omega t)$, where $\omega/k = c$.

For transverse waves on a string, the velocity of a traveling wave solution is $c = \pm\sqrt{\tau/\lambda_0}$, the kinetic energy density is $K(x,t) = (\lambda_0/2)\left(\partial\eta/\partial t\right)^2$, the potential energy density is $V(x,t) = (\tau/2)\left(\partial\eta/\partial x\right)^2$, the power is $P(x,t) = -\tau(\partial\eta/\partial t)(\partial\eta/\partial x)$ and the momentum density is $g_x(x,t) = -\lambda_0(\partial\eta/\partial t)(\partial\eta/\partial x)$.

For transverse *traveling* waves on a string, of the form $\eta(x,t) = f(x \pm ct)$, $\partial\eta/\partial x = \pm(1/c)(\partial\eta/\partial t)$. Also, the total energy density can be simplified to $E(x,t) = \tau\left(\partial\eta/\partial x\right)^2$ for such traveling waves.

For sound waves in gases and liquids, the velocity of sound is $\sqrt{B/\rho}$. The velocity of sound in air is about 340 m/s. The pressure deviation from ambient, p, is related to the displacement field s by $p = -B(\partial s/\partial x)$, and $\partial^2 s/\partial t^2 = -(1/\rho)\partial p/\partial x$.

General Solutions. The general solution to the wave equation on a string of length L with fixed boundary conditions is

$$\eta(x,t) = \sum_{n=1}^{\infty} \sin(k_n x)\,(a_n \cos(\omega_n t) + b_n \sin(\omega_n t))$$

where $\omega_n = ck_n$ and $k_n = n\pi/L$, ($n = 1$, 2 ,...). The a_n are appropriate Fourier coefficients of the initial displacements, and $\omega_n b_n$ are the Fourier coefficients of the initial velocities. The general solution on an infinite string can be written as a sum of two traveling waves

$$\eta(x,t) = f(x - ct) + g(x + ct),$$

where $f(x) + g(x)$ gives the initial displacements and $c(g'(x) - f'(x))$ gives the initial velocities.

Boundary Conditions. Fixed boundary conditions hold the displacements fixed: $\eta = \partial\eta/\partial t = 0$. Free boundary conditions have no net force at the boundary: $\partial\eta/\partial x = 0$. Fixed boundary conditions and free boundary conditions for standing waves have $\omega_n = 2\pi nc/2L$; mixed boundary conditions have $\omega_n = 2\pi(2n-1)c/4L$.

Symmetries. The wave equation is symmetric under (a) Reflection along x, $\tilde{\eta}(x,t) = \eta(-x,t)$, (b) Time reversal, $\tilde{\eta}(x,t) = \eta(x,-t)$, (c) Reflection along y, $\tilde{\eta}(x,t) = -\eta(x,t)$. It is also (d) Homogeneous, $\tilde{\eta}(x,t) = \eta(x-\Delta,t)$, (e) Time independent, $\tilde{\eta}(x,t) = \eta(x,t-\Delta)$, and invariant under (f) Sideways motion, $\tilde{\eta}(x,t) = \eta(x,t) + \Delta$.

Reflection and Transmission. The reflection and transmission coefficients $R = (Z_1 - Z_2)/(Z_1 + Z_2)$ and $T = 2Z_1/(Z_1 + Z_2)$ give the ratio of the incident and final amplitudes of waves going from one string to another, when the two strings are tied by a massless knot. The impedances $Z_i = \sqrt{\lambda_i \tau_i}$.

Group and Phase Velocities. A one dimensional chain of balls and springs with spring constant K and mass M separated by a distance a has traveling wave solutions $\exp(kx - \omega_k t)$ with dispersion relation $\omega_k = \sqrt{K/M}\sqrt{2 - 2\cos(ka)}$. The phase velocity of a wave with dispersion relation $\omega(k)$ is $\omega(k)/k$; the group velocity is $d\omega/dk$.