

NAME: _____

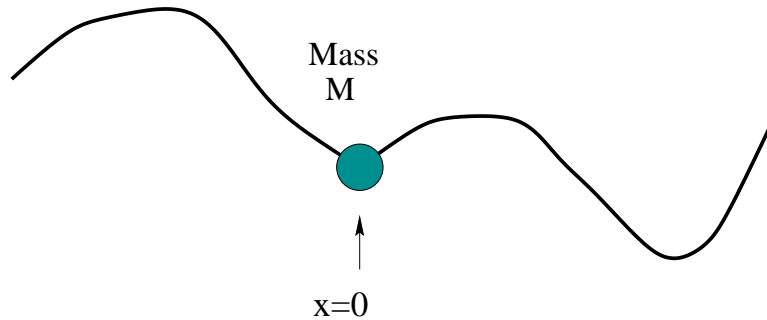
Multiple Choice (50 pts): _____ x 5pts = _____

Short Answer: S1 (50 pts) _____

TOTAL _____

**Multiple Choice: Be sure to put answers in boxes provided.
(Sorry: no partial credit!)**

M1. (10 pts) Bead on a String.

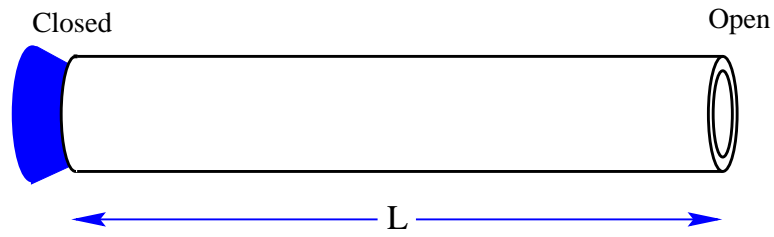


A bead of mass M is attached at $x = 0$ to a string of mass density λ_0 stretched with horizontal tension τ . The string extending to the left (negative x) has height $\eta_1(x, t)$, and to the right it has height $\eta_2(x, t)$. The bead has height $y(t) = \eta_1(0, t) = \eta_2(0, t)$. What is the formula for the acceleration of the bead? (You may wish to draw a free body diagram.)

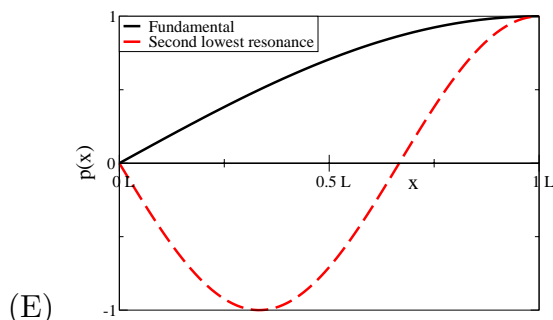
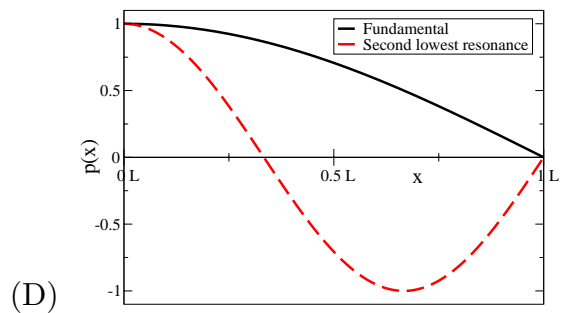
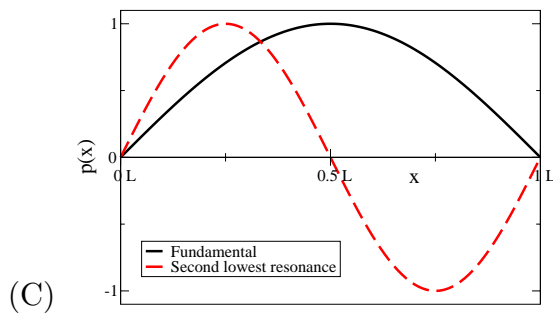
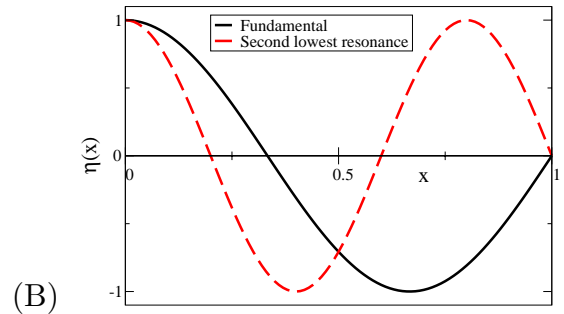
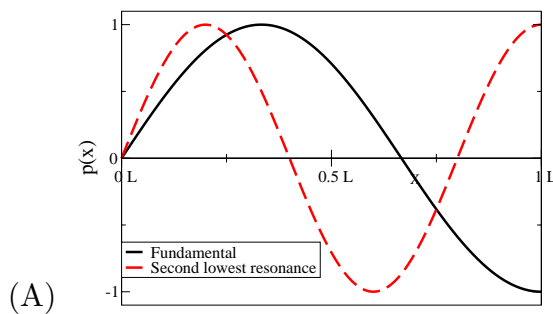
- (A) $d^2y/dt^2 = (\tau/M)\partial^2\eta/\partial x^2$
- (B) $d^2y/dt^2 = (\tau/M)(\eta_2(0, t) - \eta_1(0, t))$
- (C) $d^2y/dt^2 = (\tau/M)(\eta_1(0, t) - \eta_2(0, t))$
- (D) $d^2y/dt^2 = (\tau/M) \left(\frac{\partial\eta_2}{\partial x} \Big|_{x=0} - \frac{\partial\eta_1}{\partial x} \Big|_{x=0} \right)$
- (E) $d^2y/dt^2 = (\lambda_0/2\tau) \left(\frac{\partial^2\eta_1}{\partial x^2} \Big|_{x=0^-} + \frac{\partial^2\eta_2}{\partial x^2} \Big|_{x=0^+} \right)$

Answer

M2. (10 pts) Air in a tube.



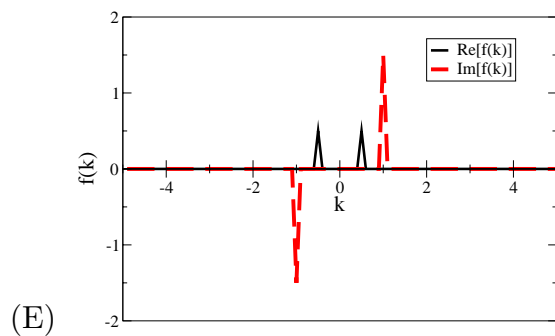
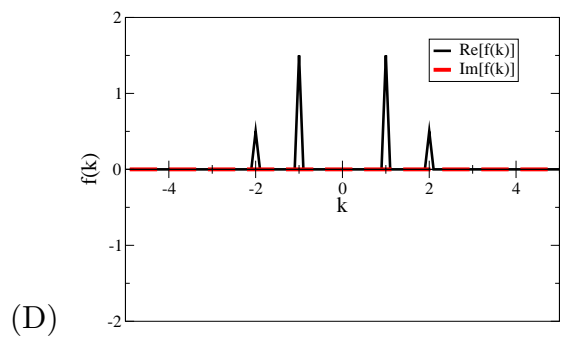
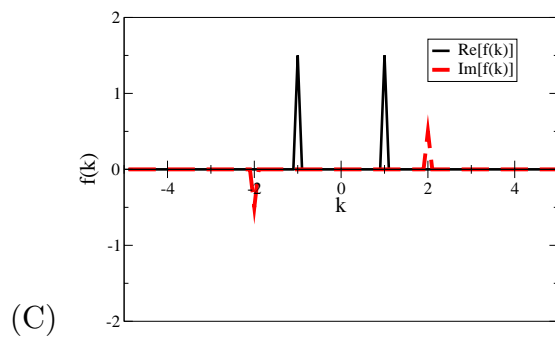
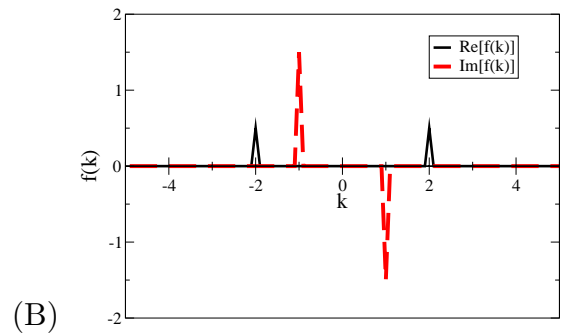
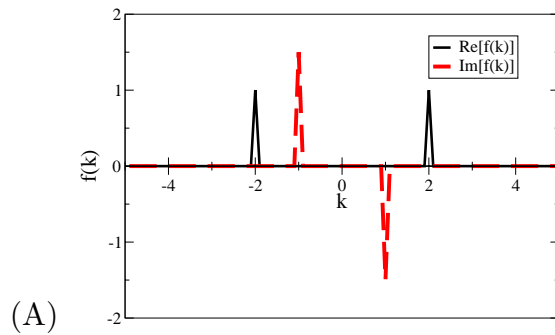
A tube of air of length L is closed on the left-hand side and open on the right. Which pictures represent the *pressure* change $p(x)$ from atmospheric pressure for the lowest two resonant frequencies in this tube? (The solid line is the fundamental, the dashed line represents the second lowest tone.)



(E)
Answer

M3. (10 pts) Fourier Series.

Which picture represents the Fourier series associated with the function $f(x) = 3 \sin(x) + \cos(2x)$? (The solid line is the real part, the dashed line is the imaginary part.)



Answer

M4. (10 pts) Wave equation with gravity.

The wave equation for a string of mass λ_0 and tension τ in a gravitational field of acceleration $g = 10m/s^2$ is

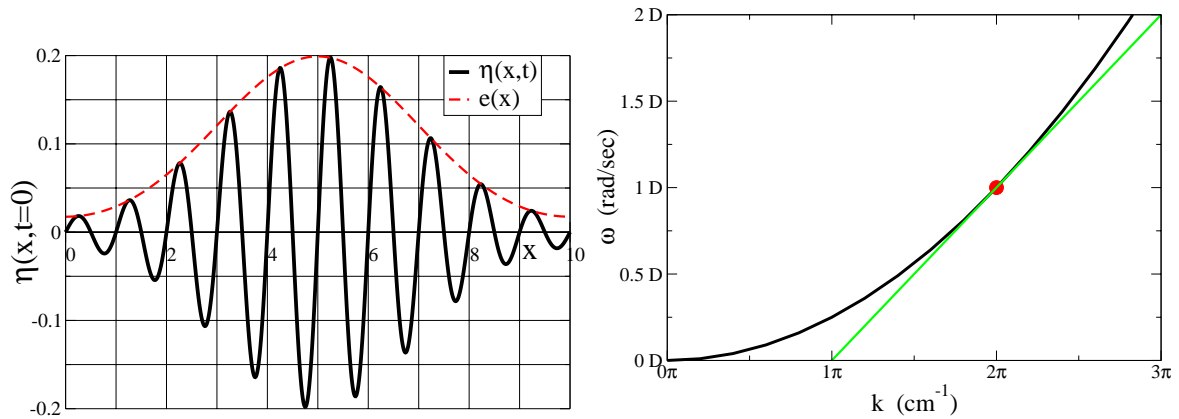
$$\partial^2\eta/\partial t^2 = (\tau/\lambda_0)\partial^2\eta/\partial x^2 - g.$$

What symmetry preserved by the wave equation is broken by the gravitational field?

- (A) Reflection along x , $\tilde{\eta}(x, t) = \eta(-x, t)$.
- (B) Time reversal, $\tilde{\eta}(x, t) = \eta(x, -t)$.
- (C) Reflection along y , $\tilde{\eta}(x, t) = -\eta(x, t)$.
- (D) Homogeneous, $\tilde{\eta}(x, t) = \eta(x - \Delta, t)$.
- (E) Time independent, $\tilde{\eta}(x, t) = \eta(x, t - \Delta)$.
- (F) Sideways motion, $\tilde{\eta}(x, t) = \eta(x, t) + \Delta$.

Answer

M5. (10 pts) Group and Phase Velocities.



A wave packet in an unusual optical fiber is traveling to the right. The wave packet has an envelope of shape $e(x)$ at time $t = 0$, and has a carrier wave with nodes at $x = 0.5$ cm, $x = 1$ cm, $x = 1.5$ cm, ... as shown. The dispersion relation for the glass in this fiber is $\omega(k) = Dk^2$, as shown.* Which of the following formulas most closely describes $\eta(x, t)$, ignoring the slow spreading of the wave packet?

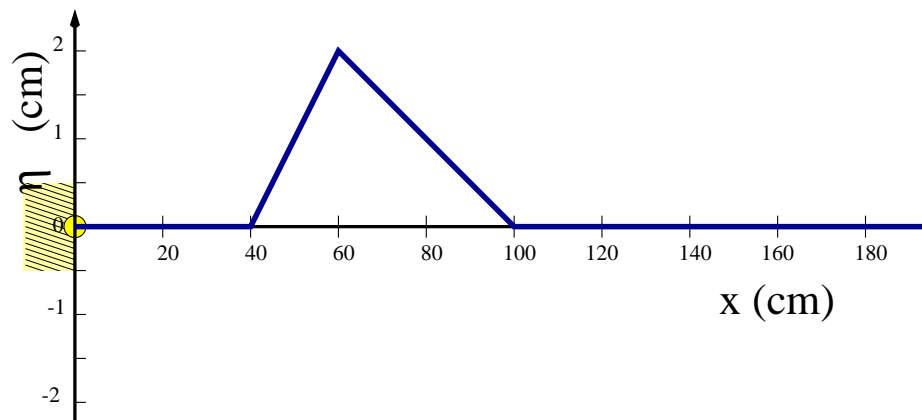
- (A) $\eta(x, t) \approx e(x - 4\pi Dt) \sin(2\pi(x - 2\pi Dt))$.
- (B) $\eta(x, t) \approx e(x - 2\pi Dt) \sin(2\pi(x - 1\pi Dt))$.
- (C) $\eta(x, t) \approx e(x - 2\pi Dt) \sin(2\pi(x - 2\pi Dt))$.
- (D) $\eta(x, t) \approx e(x - 2\pi Dt) \sin(2\pi(x - 4\pi Dt))$.
- (E) $\eta(x, t) \approx e(x - 1\pi Dt) \sin(2\pi(x - 2\pi Dt))$.
- (F) $\eta(x, t) \approx \exp(ikx - \omega t)$.

Answer

* It's not useful for this problem, but this is the dispersion relation for the wave equation $\partial^2 \eta / \partial t^2 = -D \partial^4 \eta / \partial x^4$.

Short Answer: Show Your Work

S1. (50 pts) Triangular waves on strings.



A string of mass density $\lambda_0 = 2\text{gm/cm}$ and tension $\tau = 8$ dynes starts *at rest*, displaced with a triangular form $\eta(x, 0) = f(x)$ as shown. There are fixed boundary conditions at $x = 0$.

(A) (15 points)

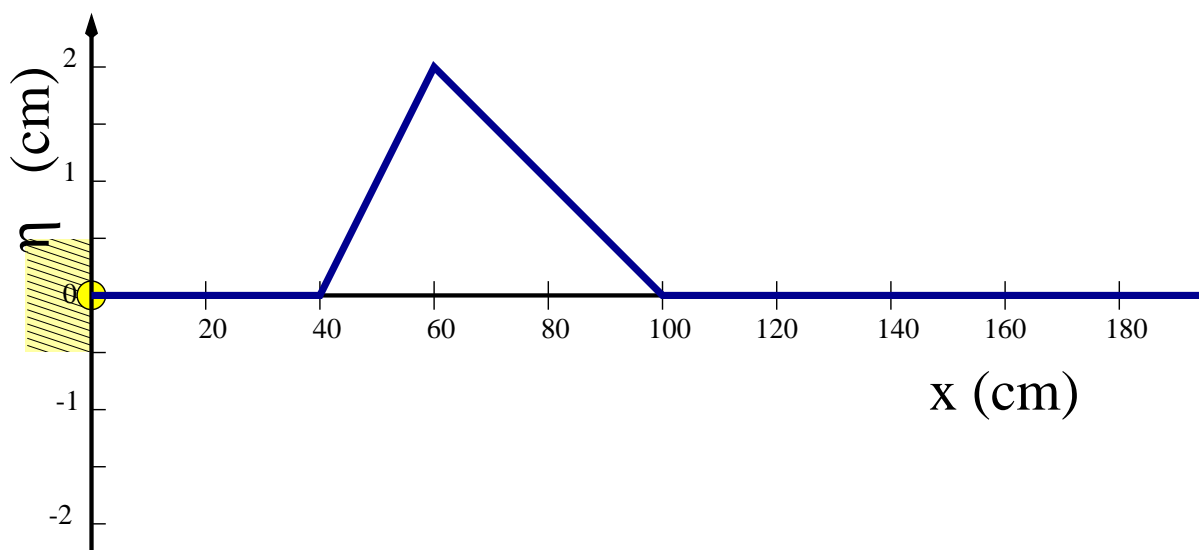
What is the total kinetic energy for the pulse? The total potential energy (integrated over x)?

Total kinetic energy: _____ ergs

Total potential energy: _____ ergs

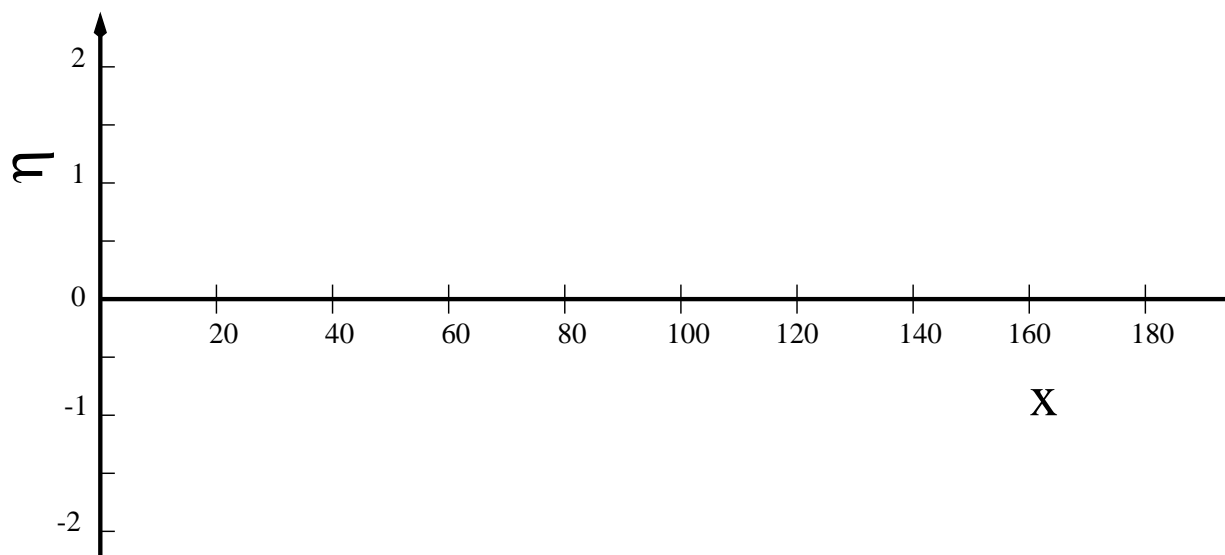
(Did you remember that the string starts at rest?)

(B) (15 points)

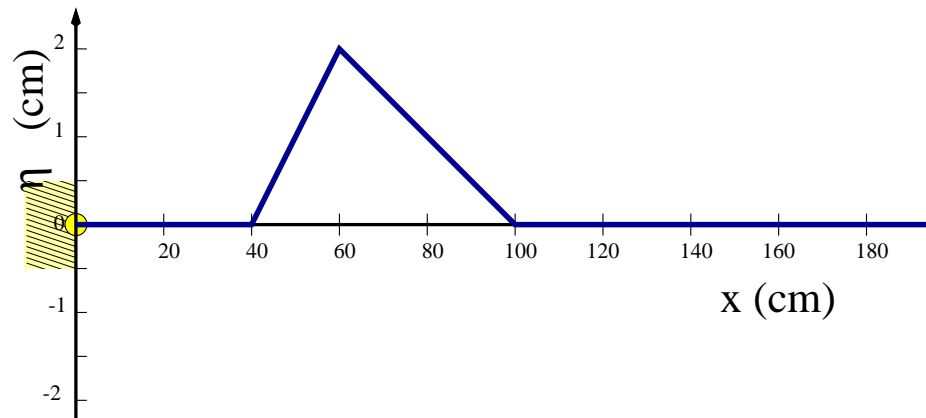


For your convenience, the triangular pulse is redrawn above; as before, the mass density is $\lambda_0 = 2$ gm/cm and tension $\tau = 8$ dynes, and there are fixed boundary conditions at $x = 0$. The string extends forever to the right.

On the graph below, draw a picture of the string at time $t = 50$ seconds.



(C) (20 points)



If the pulse shape at $t = 0$ is $\eta(x, 0) = f(x)$, give a formula for $\eta(x, t)$, valid at all times t in terms of f and the speed of transverse waves c on the string, incorporating both the initial condition at rest and the fixed boundary condition at $x = 0$.

$$\eta(x, t) = \underline{\hspace{15cm}}$$

Physics 218—Exam I (October 5)

Formula Sheet

Fall 2001

James P. Sethna

Trigonometry. The angle addition formulas are $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and $\cos(A + B) = \cos A \cos B - \sin A \sin B$. For a sine wave, $f = \omega/2\pi$, and $k = 2\pi/\lambda$. The trig functions are related to complex exponentials: $\exp(iz) = \cos(z) + i \sin(z)$, $\cos(z) = (\exp(iz) + \exp(-iz))/2$, and $\sin(z) = (\exp(iz) - \exp(-iz))/(2i)$.

Orthonormality. $(2/\pi) \int_0^\pi \sin(m\theta) \sin(n\theta) = \delta_{mn}$, $(2/\pi) \int_0^\pi \cos(m\theta) \cos(n\theta) = \delta_{mn}$, $(2/\pi) \int_0^\pi \cos(m\theta) \sin(n\theta) = 0$, and $(1/2\pi) \int_0^{2\pi} \exp(im\theta) \exp(-in\theta) = \delta_{mn}$ where $\delta_{mn} = 0$ for $m \neq n$ and $\delta_{mn} = 1$ for $m = n$.

Fourier. The formula for the Fourier series coefficients $\tilde{f}(k_m)$ of a function $f(x)$ in an interval of length L is

$$\tilde{f}^{analytic}(k_m) = (1/L) \int_{-L/2}^{L/2} f(x) \exp(-ik_m x) dx.$$

where $k_m = 2\pi m/L$. The Fourier series can be summed to retrieve the original function, just as a vector can be reconstructed by multiplying its coefficients times the basis vectors:

$$f(x) = \sum_m \tilde{f}^{analytic}(k_m) \exp(ik_m x).$$

The Fourier transform is roughly the Fourier series as $L \rightarrow \infty$, apart from normalization. The Fourier transform of a Gaussian of width σ , $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-x^2/2\sigma^2)$ is a Gaussian $\tilde{f}(k) = \exp(-\sigma^2 k^2/2)$ of width $1/\sigma$. The Fourier transform of a shifted function $f(x - \Delta)$ is $\exp(-i\Delta k) \tilde{f}(k)$.

Wave Equations. The wave equation

$$\partial^2 \eta / \partial t^2 = c^2 \partial^2 \eta / \partial x^2$$

has a traveling wave solution $\eta(x, t) = f(x \pm ct)$, a standing-wave solution $\eta(x, t) = A \sin(kx) \sin(\omega t)$, and (as a special case) a traveling sine wave $\eta(x, t) = A \sin(kx - \omega t)$, where $\omega/k = c$.

For transverse waves on a string, the velocity of a traveling wave solution is $c = \pm \sqrt{\tau/\lambda_0}$, the kinetic energy density is $K(x, t) = (\lambda_0/2) (\partial\eta/\partial t)^2$, the potential energy density is $V(x, t) = (\tau/2) (\partial\eta/\partial x)^2$, the power is $P(x, t) = -\tau (\partial\eta/\partial t) (\partial\eta/\partial x)$ and the momentum density is $g_x(x, t) = -\lambda_0 (\partial\eta/\partial t) (\partial\eta/\partial x)$.

For transverse *traveling* waves on a string, of the form $\eta(x, t) = f(x \pm ct)$, $\partial\eta/\partial x = \pm(1/c) (\partial\eta/\partial t)$. Also, the total energy density can be simplified to $E(x, t) = \tau (\partial\eta/\partial x)^2$ for such traveling waves.

For sound waves in gases and liquids, the velocity of sound is $\sqrt{B/\rho}$. The velocity of sound in air is about 340 m/s. The pressure deviation from ambient, p , is related to the displacement field s by $p = -B(\partial s/\partial x)$, and $\partial^2 s/\partial t^2 = -(1/\rho) \partial p/\partial x$.

General Solutions. The general solution to the wave equation on a string of length L with fixed boundary conditions is

$$\eta(x, t) = \sum_{n=1}^{\infty} \sin(k_n x) (a_n \cos(\omega_n t) + b_n \sin(\omega_n t))$$

where $\omega_n = ck_n$ and $k_n = n\pi/L$, ($n = 1, 2, \dots$). The a_n are appropriate Fourier coefficients of the initial displacements, and $\omega_n b_n$ are the Fourier coefficients of the initial velocities. The general solution on an infinite string can be written as a sum of two traveling waves

$$\eta(x, t) = f(x - ct) + g(x + ct),$$

where $f(x) + g(x)$ gives the initial displacements and $c(g'(x) - f'(x))$ gives the initial velocities.

Boundary Conditions. Fixed boundary conditions hold the displacements fixed: $\eta = \partial\eta/\partial t = 0$. Free boundary conditions have no net force at the boundary: $\partial\eta/\partial x = 0$. Fixed boundary conditions and free boundary conditions for standing waves have $\omega_n = 2\pi nc/2L$; mixed boundary conditions have $\omega_n = 2\pi(2n - 1)c/4L$.

Symmetries. The wave equation is symmetric under (a) Reflection along x , $\tilde{\eta}(x, t) = \eta(-x, t)$, (b) Time reversal, $\tilde{\eta}(x, t) = \eta(x, -t)$, (c) Reflection along y , $\tilde{\eta}(x, t) = -\eta(x, t)$. It is also (d) Homogeneous, $\tilde{\eta}(x, t) = \eta(x - \Delta, t)$, (e) Time independent, $\tilde{\eta}(x, t) = \eta(x, t - \Delta)$, and invariant under (f) Sideways motion, $\tilde{\eta}(x, t) = \eta(x, t) + \Delta$.

Reflection and Transmission. The reflection and transmission coefficients $R = (Z_1 - Z_2)/(Z_1 + Z_2)$ and $T = 2Z_1/(Z_1 + Z_2)$ give the ratio of the incident and final amplitudes of waves going from one string to another, when the two strings are tied by a massless knot. The impedances $Z_i = \sqrt{\lambda_i \tau_i}$.

Group and Phase Velocities. A one dimensional chain of balls and springs with spring constant K and mass M separated by a distance a has traveling wave solutions $\exp(kx - \omega_k t)$ with dispersion relation $\omega_k = \sqrt{K/M} \sqrt{2 - 2 \cos(ka)}$. The phase velocity of a wave with dispersion relation $\omega(k)$ is $\omega(k)/k$; the group velocity is $d\omega/dk$.