Physics 218—Final Exam (December 18) **Fall 2001**

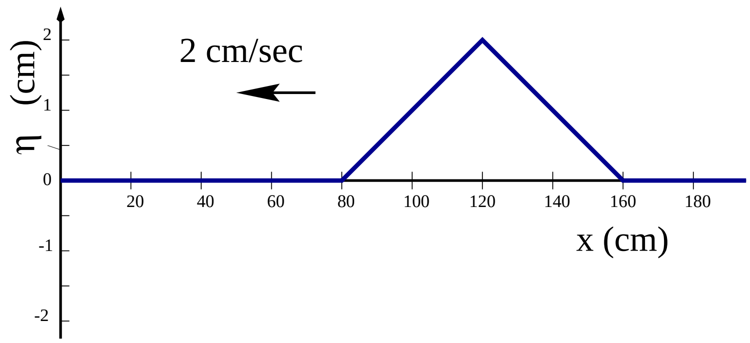
James P. Sethna

NAME:________________________________

Multiple Choice (60 pts):_____	**x 10pts =**	____________________
Short Answer:	**S1 (40 pts)**	____________________
	S2 (30 pts)	____________________
	S3 (25 pts)	____________________
	S4 (45 pts)	____________________
	TOTAL (200 pts)	____________________

Multiple Choice: Be sure to put answers in boxes provided. (Sorry: no partial credit!)

M1. (10 pts) Energy and Power: Waves on a String.



A triangular transverse pulse is traveling to the left down a string of mass density $\lambda = 2$ gm/cm and tension $\tau = 8$ dynes, with shape and dimensions as shown in the plot. Inside the pulse, what is the energy density E and the power P? What is the total energy U stored in the pulse? (Hint: before you plunge in, notice the choices below.)

(A) $E = 0.02$ ergs/cm, $P = 0.04$ ergs/second, $U = 1.6$ ergs

(B) $E = -0.02$ ergs/cm, $P = -0.04$ ergs/second, $U = 1.6$ ergs

(C) $E = 0.02$ ergs/cm, $P = -0.04$ ergs/second, $U = 1.6$ ergs

(D) $E = 0.02$ ergs/cm, $P = 0.04$ ergs/second, $U = -1.6$ ergs

(E) $E = -0.02$ ergs/cm, $P = -0.04$ ergs/second, $U = -1.6$ ergs

Answer

Related formulæ:
$E = (1/2)\tau(\partial\eta/\partial x)^2 + (1/2)\lambda_0(\partial\eta/\partial t)^2$,
$P = -\tau(\partial\eta/\partial x)(\partial\eta/\partial t)$, $U = \int E\,dx$;
for traveling waves $\partial\eta/\partial t = -v\partial\eta/\partial x$.

M2. (10 pts) Fourier Transforms.

A musical instrument playing a note of frequency ω_1 generates a pressure wave $P(t)$ periodic with period $2\pi/\omega_1$: $P(t) = P(t + 2\pi/\omega_1)$. The complex Fourier series of this wave is zero except for $n = \pm 1$, corresponding to the fundamental ω_1. At $n = 1$, the Fourier amplitude is $2 + 3i$, and at $n = -1$ it is $2 - 3i$. What is the pressure $P(t)$?

(A) $\exp\left((2 + 3i)\omega_1 t\right)$

(B) $\exp\left((4\omega_1 t)\right) \exp\left(i(3\omega_1 t)\right)$

(C) $\cos 2\omega_1 t - \sin 3\omega_1 t$

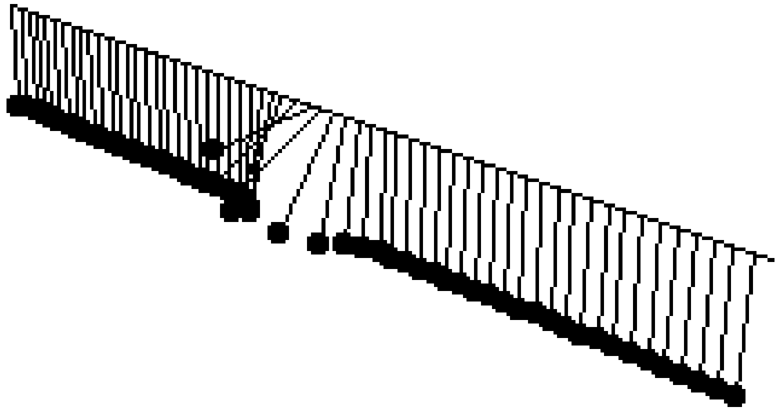
(D) $4 \cos \omega_1 t - 6 \sin \omega_1 t$

(E) $4 \cos \omega_1 t + 6 \sin \omega_1 t$

Answer

Related formulæ:
$\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$, $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$,
$f(t) = \sum_n \tilde{f}_n \exp(i\omega_n t)$,
$\tilde{f}_n = (1/T) \int_0^T f(t) \exp(-i\omega_n t)\, dt$,

M3. (10 pts) Sine-Gordon Dispersion Relation.



An array of pendula connected by springs, in the continuum limit, obeys the Sine-Gordon equation

$$\partial^2\phi/\partial t^2 = A\partial^2\phi/\partial x^2 - B\sin(\phi).$$

with $\phi(x) = 0$ corresponds to the pendulum at position x along the array pointing downward. What is the dispersion relation $\omega(k)$ for small oscillations in this equation?

(A) $\omega(k) = \left(-B \pm \sqrt{B^2 - 4Ak^2}\right)/2A$

(B) $\omega(k) = \sqrt{Ak^2 - B\sin(\phi)}$

(C) $\omega(k) = \sqrt{Ak^2 - B}$

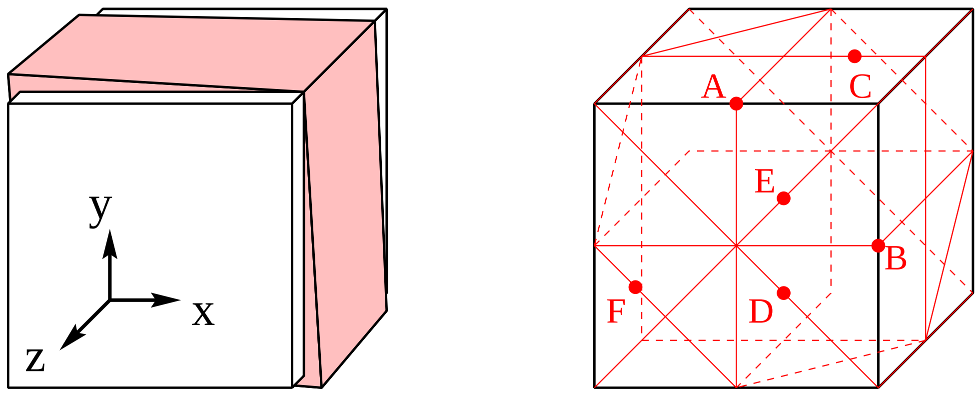
(D) $\omega(k) = \sqrt{Ak^2 + B}$

(E) $\omega(k) = \sqrt{Ak^2 + B\sin(\phi)}$

Answer

Related formulæ:
For small θ, $\sin\theta \sim \theta$
Plane waves: $e^{i(kx-\omega(k)t)}$, $\cos(kx - \omega(k)t)$, $\sin(kx - \omega(k)t)$
Phase velocity $\omega(k)/k$, Group velocity $d\omega/dk$.

M4. (10 pts) Tensors.



An isotropic elastic medium is strained as shown on the left above: it is compressed and stretched along different axes. The stress tensor is

$$\sigma_{ij} = \begin{pmatrix} a & -a & 0 \\ -a & a & 0 \\ 0 & 0 & -2a \end{pmatrix}.$$

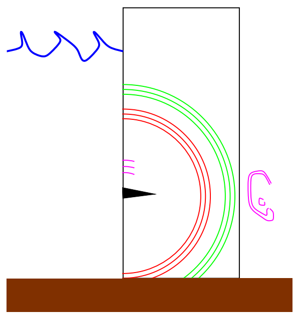
The medium has a flat free surface perpendicular to the axis $\hat{\mathbf{n}}$. (A free surface is a surface on which there is no traction, or forces, applied.) Knowing the stress tensor above, in which direction $\hat{\mathbf{n}}$ could the surface normal point? The surfaces are illustrated in the figure on the right.

(A) $\hat{\mathbf{n}} = (1, 0, 0)$
(B) $\hat{\mathbf{n}} = (0, 1, 0)$
(C) $\hat{\mathbf{n}} = (0, 0, 1)$
(D) $\hat{\mathbf{n}} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$
(E) $\hat{\mathbf{n}} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$
(F) $\hat{\mathbf{n}} = (1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$

Answer

Related formulæ:
$F_i/A = \sigma_{ij}\hat{\mathbf{n}}_j$
$F_i = \partial_j \sigma_{ij}$
$\sigma_{ij} = c_{ijkl}\epsilon_{kl} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$

M5. (10 pts) Elastic Waves.



A small crack starts on the inside of a concrete dam, generating acoustic waves of all polarizations with wavelengths much shorter than the thickness D of the dam. An acoustical detector is positioned outside the dam directly opposite to the crack. The concrete can be assumed to be an isotropic medium with positive elastic constants λ and μ. What signal is expected in the acoustical detector?

(A) A transverse sound pulse, followed by a longitudinal sound pulse.

(B) A longitudinal sound pulse, followed by a transverse sound pulse.

(C) A transverse sound pulse only: sound is a transverse wave.

(D) A longitudinal sound pulse only: the transverse sound component will travel along the length and width of the dam, not across the thickness.

(E) A sound pulse after a time $t = D/\sqrt{Y/\rho}$, where Y is the Young's modulus of concrete.

Answer

Related formulæ:
$\rho \partial^2 u_i/\partial t^2 = (\lambda + \mu)\partial_i \partial_j u_j + \mu \partial_j \partial_j u_i$,
With $\nabla \cdot \mathbf{u}_T = 0$ and $\nabla \times \mathbf{u}_L = 0$,
$c_T = \sqrt{\mu/\rho}$ and $c_L = \sqrt{(\lambda + 2\mu)/\rho}$

M6. (10 pts) Molecular Motors and Random Walks.

A molecular motor has a stepsize d between local minima of its Gibbs free energy. An external force F is applied which is just sufficient to halt all average motion, making the different minima equal in free energy. The barrier to moving between different minima B is thus the same for forward and backward hops. Which expression is expected for the root-mean-square distance $R(t) = \sqrt{\langle (x(t) - x(0))^2 \rangle}$ that the motor will diffuse due to random hops?

(A) $R(t) = (F/m)t^2$

(B) $R(t) = d\sqrt{t\Gamma_0 \exp(-B/k_B T)}$

(C) $R(t) = d\sqrt{\exp(B/k_B T)/\Gamma_0 t}$

(D) $R(t) = vt \exp(-B/k_B T)$

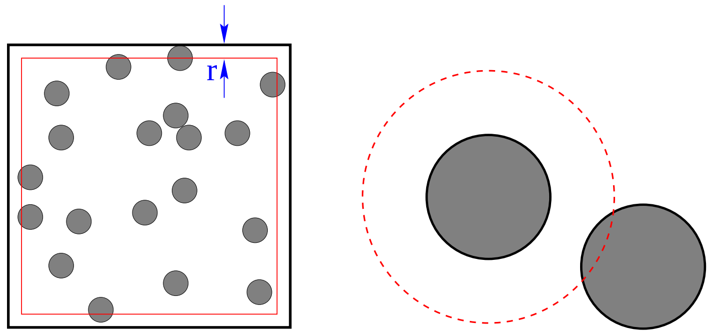
Answer

Related formulæ:

$\langle (x - x_0)^2 \rangle \sim N\sigma^2$

$\Gamma = \Gamma_0 \exp(-B/k_B T)$

Short Answer: Show Your Work
S1. (40 pts) Entropy and Hard Spheres.



We can improve on the realism of the ideal gas by giving the atoms a small radius. If we make the potential energy infinite inside this radius ("hard spheres"), the potential energy is simple (zero unless the spheres overlap, which is forbidden). Let's do this in two dimensions.
A two dimensional $L \times L$ box contains an ideal gas of N hard disks of radius $r \ll L$ (left figure). The disks are dilute: the summed area $N\pi r^2 \ll L^2$. Since the disks cannot be within r of the edges of the box, let A be the effective volume allowed for the first disk in the box: $A = (L - 2r)^2$.

(S1.A) (10 pts) **Configuration Space Volume for Hard Disks.**
The area allowed for the second disk is $A - \pi(2r)^2$ (right figure), ignoring the small correction when the excluded region around the first disk overlaps the excluded region near the walls of the box. The area allowed for the n^{th} disk is $A - (n-1)\pi(2r)^2$, ignoring corrections for the overlaps of the excluded regions. Let configuration space $\mathbf{X}$ be the $2N$ dimensional space of positions $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\ldots \mathbf{x}^{(N)}$. Write an expression for the volume $\Omega_\mathbf{X}$ of allowed zero-energy configurations of hard disks, in the configuration space $\mathbf{X}$, ignoring the overlapping excluded regions.

$\Omega_\mathbf{X} =$ ______________________

Related formulæ: For a 3D ideal gas, $\Omega_\mathbf{P} = (\pi' s)(2mE)^{(3N-1)/2}$, $\Omega_\mathbf{X} = V^N$.

(S1.B) (10 pts) **Statistical Mechanical Entropy for Hard Disks.**
It's now easy to write the configurational entropy, $S_{\mathbf{X}}$ for the hard disks of part (S1.A) as a sum over n. Use the "Math truth" below to find a formula for the entropy that does not involve a sum over n, accurate to first order in the area of the disks πr^2.

$S_{\mathbf{X}} =$________________________

Related formulæ: $S = k_B \ln(\Omega)$
Math Truth: To first order in ϵ, $\sum_{n=1}^{N} \log\left(A - (n-1)\epsilon\right) = N \log\left(A - (N-1)\epsilon/2\right)$.

(S1.C) (20 pts) **Pressure for Hard Disks.**
Assume the hard-disk configurational entropy is $S_{\mathbf{X}} = Nk_B \log(A - Nb)$ for some area b, representing the effective excluded area due to the other disks. (Your answer to (S1.B) won't quite have this form, but it's a good approximation.) Just as for the ideal gas, the internal energy is purely kinetic, and the kinetic energy and momentum-space entropy depend only on temperature and not on volume. So, if we isothermally expand this hard-disk gas from initial area A_1 to A_2, the heat flow Q is both related to the entropy and to the pressure. By differentiating with respect to A_2, find the pressure for the hard-sphere gas. (Hint: for $b = 0$ it should reduce to the ideal gas law.)

$P =$ ______________________

Related formulæ: $Q = \int_{A_1}^{A_2} P dA$ (First law) and $\Delta S = Q/T$ (Thermodynamic Entropy). For a 3D ideal gas, $PV = Nk_BT$ and $U = 3/2\, Nk_BT$.

S2. (30 pts) Heat Diffusion Spot.

The diffusion equation for the heat density in a two-dimensional sheet is

$$\partial q/\partial t = D(\partial^2 q/\partial x^2 + \partial^2 q/\partial y^2).$$

(S2.A) (15 pts) **Diffusion in Two Dimensions.** Show that if $f(x,t)$ satisfies the diffusion equation in one dimension, then $f(x,t)f(y,t)$ solves the diffusion equation in two dimensions.

Related formulæ: Product Rule, $\partial fg/\partial z = \partial f/\partial z\, g + f\, \partial g/\partial z$

(S2.B) (15 pts) **The heat spot.**
A screen of thermal diffusion constant D is heated at $x = y = 0$ and $t = 0$ by a thin laser beam pulse. The total heat deposited is Q. Use part (A) and the Greens function for the one dimensional diffusion equation to derive the equation for $q(x, y, t)$, the heat density after a time t. What is the root-mean-square radius $r(t) = \sqrt{\langle x^2 + y^2 \rangle}$ for this spot?

$q(x, y, t) =$ __________________________

$r(t) =$ __________________________

Related formulæ: $\partial\rho/\partial t = D\partial^2\rho/\partial x^2$.
If $\rho(x, 0) = \delta(x)$, $\rho(x, t) = G(x, t) = \frac{1}{\sqrt{2\pi Dt}} e^{-x^2/2Dt}$ and $\langle x^2 \rangle = Dt$.
$\langle f(\mathbf{z}) \rangle = \int f(\mathbf{z})\rho(\mathbf{z})\, d^D z$.

S3. (25 pts) Deriving New Laws.

The evolution of a physical system is described by a field Ξ, obeying a partial differential equation

$$\partial\Xi/\partial t = A\,\partial\Xi/\partial x. \qquad (S3.1)$$

(S3.A) (15 pts) **Symmetries.**
Circle ALL the symmetries that this physical system appears to have:

(A) Spatial inversion ($x \to -x$).
(B) Time reversal symmetry ($t \to -t$).
(C) Order parameter inversion $\Xi \to -\Xi$).
(D) Homogeneity in space ($x \to x + \Delta$).
(E) Time translational invariance ($t \to t + \Delta$).
(F) Order parameter shift invariance ($\Xi \to \Xi + \Delta$).

¿Did you circle ALL correct choices?

(S3.B) (10 pts) **Traveling Waves.**
Show that our equation $\partial\Xi/\partial t = A\,\partial\Xi/\partial x$ has a traveling wave solution. If $A > 0$, which directions can the waves move?

Related formulæ: Traveling Wave $f(x - vt)$

S4. (45 pts) Life and the Heat Death of the Universe.

Freeman Dyson discusses how living things might evolve to cope with the cooling and dimming we expect during the heat death of the universe.
Dyson models an intelligent being as a heat engine that consumes a fixed entropy ΔS per thought. (This correspondence of information with entropy is a standard idea from computer science.)

(S4.A) (15 pts) **Energy needed per thought.**
Assume that the being draws heat Q from a hot reservoir at T_1 and radiates it away to a cold reservoir at T_2. What is the minimum energy Q needed per thought, in terms of ΔS and T_2? (You may take T_1 very large.)

$Q \geq$ ______________________

Related formulæ: For Carnot engine, $\Delta S = Q_2/T_2 - Q_1/T_1 = 0$
First Law: $Q_1 - Q_2 = W$ (energy is conserved).

(S4.B) (10 pts) **Time needed per thought to radiate energy.**
Dyson shows, using theory not important here, that the power radiated by our intelligent–being–as–heat–engine is no larger than CT_2^3, a constant times the cube of the cold temperature.* Write an expression for the maximum rate of thoughts per unit time dH/dt (the inverse of the time Δt per thought), in terms of ΔS, C, and T_2.

$dH/dt \leq$ ____________________

* The constant scales with the number of electrons in the being, so we can think of our answer Δt as the time per thought per mole of electrons.

(S4.C) (10 pts) **Number of thoughts for an ecologically efficient being.**
Our universe is expanding: the radius R grows roughly linearly in time t. The microwave background radiation has a characteristic temperature $\Theta(t) \sim R^{-1}$ which is getting lower as the universe expands: this red-shift is due to the Doppler effect. An ecologically efficient being would naturally try to use as little heat as possible, and so wants to choose T_2 as small as possible. It cannot radiate heat at a temperature below $T_2 = \Theta(t) = A/t$. How many thoughts H can an ecologically efficient being have between now and time infinity, in terms of ΔS, C, A, and the current time t_0?

$H \leq$ ________________________

(S4.D) (10 pts) **Time without end: Greedy beings.**
Dyson would like his beings to be able to think an infinite number of thoughts before the universe ends, but consume a finite amount of energy. He proposes that his beings need to be profligate in order to get their thoughts in before the world ends: he proposes that they radiate at a temperature $T_2(t) \sim t^{-3/8}$ which falls with time, but not as fast as $\Theta(t) \sim t^{-1}$. Show that with Dyson's cooling schedule, the total number of thoughts H is infinite, but the total energy consumed U is finite.