## Elementary Scientific Mathematics Defined

What are the elementary mathematical/analytical tools essential to science and engineering? The mathematics and physics education literature discusses this extensively, and offers many approaches to teaching these tools. Motivated by this literature and by direct experience as practicing scientists, those involved in the revision of Physics 207/208 at Cornell over the last 15 years have sought to incorporate these tools in our lectures, problem assignments and labs.

The following is a list of tools that we consider essential to interpreting data and solving problems, in roughly the order they typically appear in our first-semester introductory physics courses. All but the last three topics are covered in our current implemenation of our tutorial and evaluation tool.

**Algebra.** Solving for an unknown. Solving two equations in two unknowns by substitution. Quadratic equation. Multiplying by common factors. Reducing to a common denominator. Algebra of equations with more than one independent variable.

**Properties and sketching of simple functions.** E.g, x, $x^2$, $x^3$; ax+b, $ax+bx^2+cx^3$, 1/x, $1/x^2$, 1/(1-x), $1/(1-x^2)$. Use of asymptotic behavior as $x\rightarrow 0$, $x\rightarrow\infty$, and any other special values to make approximate sketches. Sketching and limits of algebraic forms encountered in physics, eg., 1/(1+x/a), (x/b)/(1+x/a), $y(x_1,x_2) = (x_1\ x_2)/(x_1 + x_2)$, $y(x_1,x_2)=x_1+x_2$. Graphical relation between powers of x. Even and odd functions. Scaling and shifting, e.g., f(ax), f(x+a), af(x) and their graphical representation.

**Graphing and interpreting data 1.** Plotting graphs from tables of numbers. Reading values off of graphs. Linear fits. Graphs as physical maps (e.g. y(x) versus y(t)); parametric equations.

**Triangles.** Relation between interior and exterior angles, between angles and side lengths/components in terms of trig functions. Determining the appropriate trig function by considering limiting behavior of side lengths as an angle goes to zero or becomes very large. Slopes and tangents. Similar triangles. Relation to more complex geometric forms like trapezoids, hexagons, cones, etc.

**Vectors.** Graphical representation. Addition, subtraction, multiplication. Coordinate axes and representation in polar or component form. Unit vectors. Resolving vectors into components along given coordinate axes; specific examples encountered in physics (e.g., masses on inclines with friction, rope/pulley configurations).

**Proportions.** E.g., if $y = A\ x^3$, by what multiplicative factor will y change if x is doubled? Using proportions to make predictions based on data in tabular or graphical form. Use in problems with more than one independent variable. Limits on the use of proportional reasoning.

**Sines and cosines 1.** Sketching trig and inverse-trig functions. Even/odd properties. Special angles & values. Degree/radian/second conversions. Small angle approximations. Basic trig identities, including their graphical representation/verification. Solving equations for unknowns using trig and inverse trig functions and identities.

**Calculus 1.** Basic notions of integrals and derivatives. Obtaining integrals and derivatives from data in tabular or graphical form; derivative = slope of curve at a point, integral = area under curve between two points, second derivative = curvature. Graphical relations between a function and its derivatives. Average values. Numerical differentiation and integration.

**Units and dimensional analysis.** Showing that equations have consistent dimensions. Using dimensions to determine relations between variables. Unit conversions. Equivalence of units (e.g., pressure = force/area or energy/volume). Dimensionless combinations of variables. Writing equations in dimensionless form. Advantages of graphical representations using dimensionless variables.

**Orders of magnitude.** What is an "order of magnitude"? An "order of magnitude" estimate? What are for the relative magnitudes of physical quantities (e.g., speeds of objects in m/s)? "Back of the envelope" calculations.

**Calculus 2.** Formal algebraic differentiation and integration of constants, lines, powers of x, sines and cosines, exponentials and logs. Definite integrals of common functions. Effects of scaling, e.g., $f(x) \rightarrow f(ax)$ on integrals and derivatives.

**Sines and cosines 2.** $A \sin(\omega t + \phi)$ and $Ae^{-t/\tau} \sin(\omega t + \phi)$ - the mathematics of oscillations. How changing A, $\omega$, $\phi$ and $\tau$ affects graphical representations. Graphical integration and differentiation.

**Small changes.** $(1+x)^n$ for small x. Transforming expressions to this standard form. Percentage changes and their effects. Key concepts of Taylor series, including graphical interpretation. Linearization.

**Geometry.** Areas of rectangles, triangles, trapezoids and circles. Volumes of cubes, pyramids, spheres and cylinders, cones. Surface to volume ratios; circumference to area ratios; limiting behaviors. Scaling of geometric quantities with a characteristic length.

**Exponentials.** $\exp(t/\tau)$, $\exp(-x/\lambda)$. Time constants, scale lengths. Doubling time, decade time, half-life. Changing bases. Identities. Graphing; approximate sketches using $t/\tau=1$, $\rightarrow 0$, $\rightarrow \infty$, $\rightarrow -\infty$. Does an exponential increase faster than x? $x^2$? $x^3$? Does proportional reasoning apply? What do "exponentially increasing" and "exponentially decreasing" mean? Relation to geometric series.

**Logs.** Basic relations/identities and graphing. Numerical relation between a value and its log. Changing bases. Using logs to solve for unknowns.

**Semi-log plots.** Generating semi-log plots from a data set. What does it mean to label the log axis with the values corresponding to the argument of the log? Using semi-log plots to determine if a given data set has an exponential dependence. Determining the numerical value of the slope from graphs labeled with log(y) and y values, and extracting the time constant and prefactor.

**Log-log plots.** Generating log-log plots using a given data set. Implications of labeling the axes with the arguments of the logs, not the values of the logs themselves. Using log-log plots to determine power-law relations between x and y, e.g., $y = A x^3$. Independence of slope on choice of base for log. Allometry and scaling.

**Generalized graphical analysis.** Choosing the quantities plotted so that data obeying a particular equation will appear as a straight line.

**Sines and cosines 3.** Traveling waves ,$A\cos(kx - \omega t + \phi)$. Interpreting and representing functions of a difference between two variables. Relation between periodicities in time and space. Addition/subtraction of traveling waves.

**Calculus 3.** Newton's equation; harmonic oscillator; damped harmonic oscillator; diffusion equation; wave equation.

**Elements of Fourier analysis.** Orthogonal functions. Representing complex periodic and nonperiodic functions in 1D and 2D using sums of sinusoids. Ideas of a spectrum and of frequency/wavenumber space. Importance of amplitudes and phases in determining the synthesized waveform.

**Probability.** Random numbers. Gaussian, binomial, and Poisson distributions. Brownian motion.