ASTR415: Problem Set #5

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Three systems of coupled differential equations were studied using integrators based on Euler's method, a fourth-order Runge-Kutta method, and the Leapfrog method. For even a basic system of two smooth equations, Euler's method performed very poorly and was slow to converge, as the error scaled as $O(h)$. The Runge-Kutta method performed well for all three systems, converging quickly even for moderate time steps and exhibiting an error that scaled as $O(h^4)$. The Leapfrog method, where applicable, also converged quickly to a solution, although the error only scaled as $O(h^2)$. The leapfrog method proved to be the most stable, conserving the energy of an orbit throughout a long integration. Finally, a bisection technique was used when studying a Lotka-Volterra Predator-Prey model to determine that the minimum hunting rate to achieve extinction was $q = 1.24$.

I. A ONE-DIMENSIONAL SECOND-ORDER ODE

Consider the second-order ordinary differential equation

$$\frac{d^2x}{dt^2} = -x \tag{1}$$

with the initial conditions $x(0) = 0$, $\dot{x} = 1$. The solution is given by $x(t) = \sin(t)$. Note that this is a conservative equation, since the second derivative of x is independent of t and $\dot{x}$. Therefore, the Leapfrog method is a valid integration scheme.

This second-order equation can be written as a system of the coupled first-order equations

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x \end{aligned} \tag{2}$$

with initial conditions $x(0) = 0$, $y(0) = 1$. Here, the exact solution for y is given by $y(t) = \cos(t)$. This system is in a form that can be integrated using Euler's method or a Runge-Kutta method.

Euler's method, the fourth-order Runge-Kutta method (as described in [1]), and the Leapfrog method can all be used with a constant step size h. When solving this equation numerically, h was chosen to be 1, 0.3, 0.1, 0.03, and 0.01, and the equation was integrated from $t = 0$ to $t = 15$, representing about 2.5 periods of the sine wave. The results from Euler's method are shown in figure 1, the results from the fourth-order Runge-Kutta method are shown in figure 2, and the results from the Leapfrog method are shown in figure 3. In all cases, the exact solution $x(t) = \sin(t)$ is plotted as a dashed curve.

Euler's method converges to the solution very slowly and cannot maintain a constant amplitude even with small values of h. The Runge-Kutta method does a much better job at approximating the solution, being visually indistinguishable from the exact solution as early as $h = 0.1$. Furthermore, the amplitude of its solutions does not appear to grow over time on this time scale. Finally, the Leapfrog solutions also converge very quickly, being visually identical to the exact solution by $h = 0.1$ and maintaining a constant amplitude. However, for the very large time step $h = 1$, the solution drifts out of phase with the exact solution fairly quickly. This behavior was not observed for the Runge-Kutta solution using the same time step.

For each of these methods, we expect the error at any time in the solution to decrease as some power of the step size h. To confirm this behavior, $|x(15) - \sin(15)|$ is plotted against h on a logarithmic scale in figure 4. For all three methods, the slope is constant for $h \leq 0.1$. In particular, the slope for Euler's method is 1.20, the slope for the fourth-order Runge-Kutta method is 3.92, and the slope for the Leapfrog method is 2.00. From this, we can conclude that the error in the integration is $\propto h$ for Euler's method, $\propto h^4$ for the Runge-Kutta method, and $\propto h^2$ for the Leapfrog method.

These are the expected dependencies for the error for each of the methods. Euler's method is a first-order method, meaning that the error in each step is $O(h^2)$. Since the error at t_f is the accumulation of the error from $(t_f - t_0)/h$ steps, this final error is $O(h)$, as was observed above. Similarly, the Runge-Kutta method used is fourth-order, with an error in each term of $O(h^5)$. Summed over all steps, the total error should be $O(h^4)$. Finally, the error in each term of the Leapfrog method is $O(h^3)$ for a total accumulated error of $O(h^2)$.

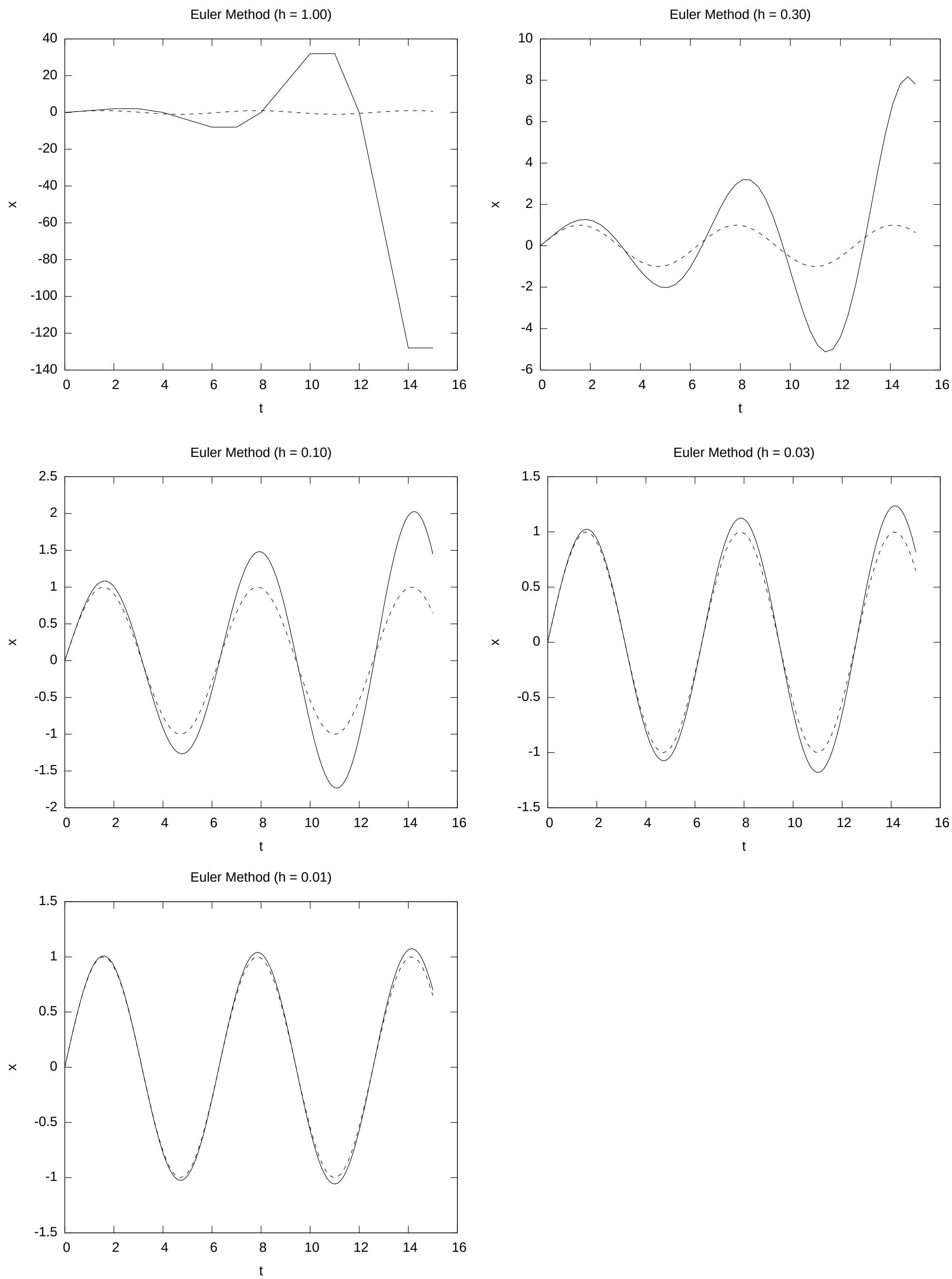

FIG. 1: Solutions of x obeying equation 2 using Euler's method for various time steps h. The exact solution is represented by the dashed curve.

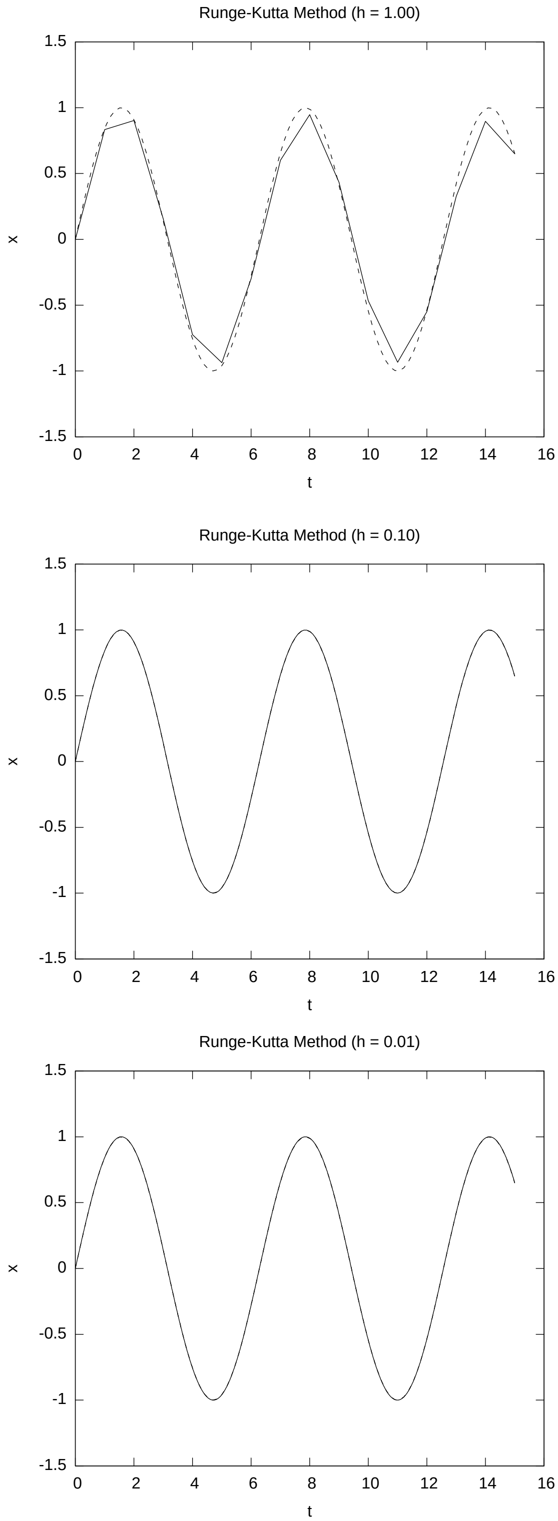

FIG. 2: Solutions of x obeying equation 2 using the fourth-order Runge-Kutta method for various time steps h. The exact solution is represented by the dashed curve.

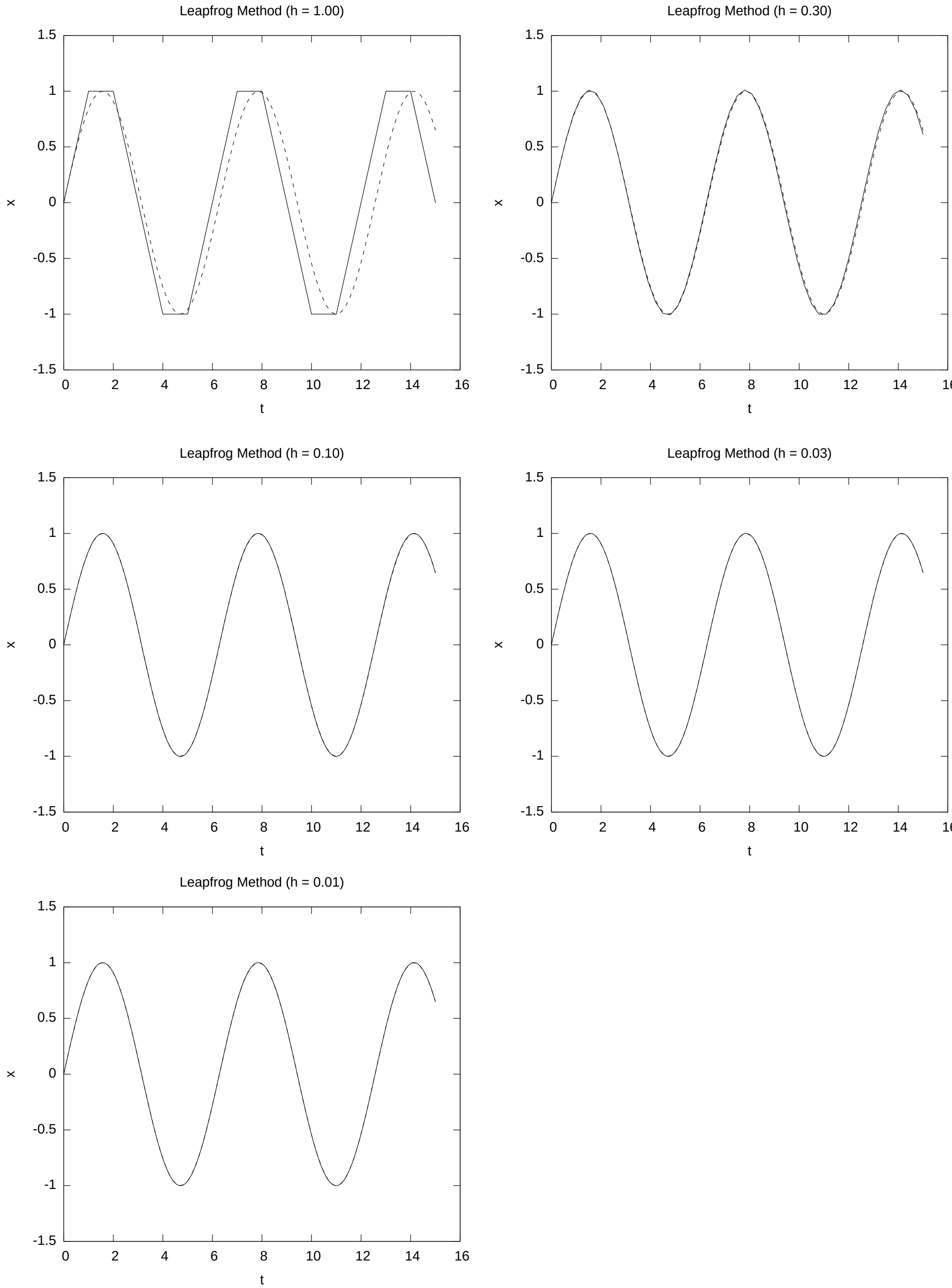

FIG. 3: Solutions of equation 1 using the Leapfrog method for various time steps h. The exact solution is represented by the dashed curve.

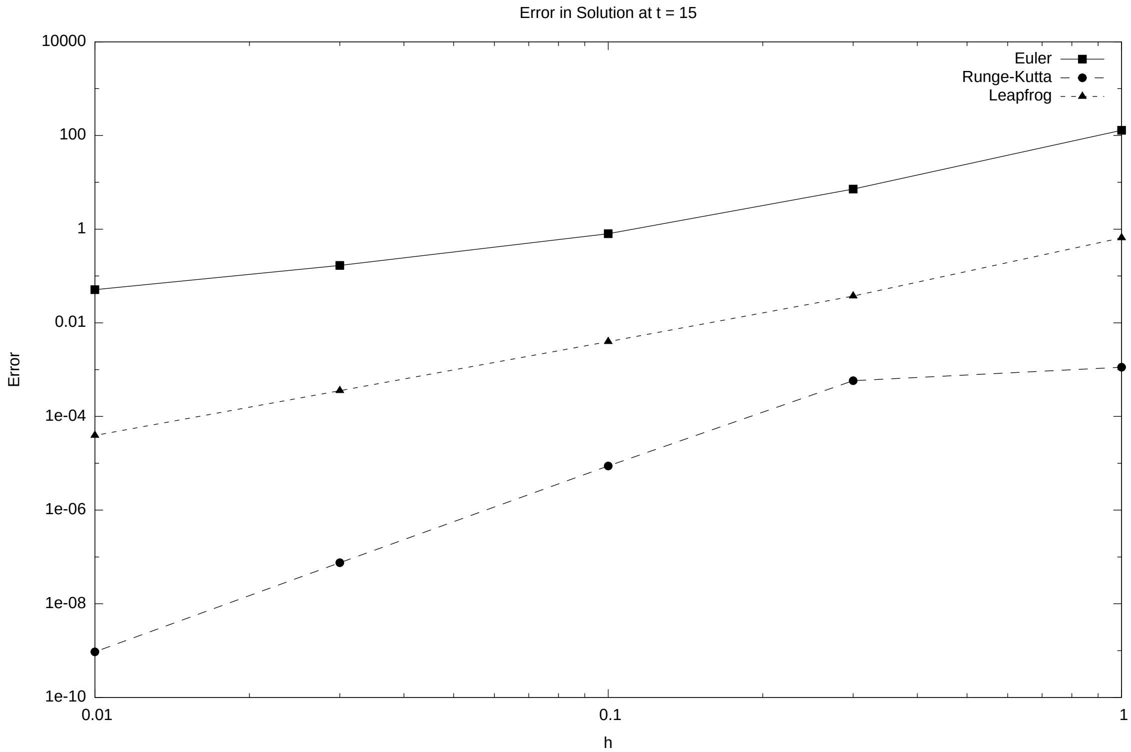

FIG. 4: The error of all three methods at $t = 15$ for each time step h.

II. A TWO-DIMENSIONAL ORBIT

Consider the two-dimensional orbit of a mass in the gravitational potential given by

$$\Phi = -(1 + 2x^2 + 2y^2)^{-1/2} \tag{3}$$

The resulting acceleration is given by $\ddot{\mathbf{x}} = -\nabla\Phi$. Writing the components explicitly,

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{d\Phi}{dx} = -2x(1 + 2x^2 + 2y^2)^{-3/2} \\ \frac{d^2y}{dt^2} &= -\frac{d\Phi}{dy} = -2y(1 + 2x^2 + 2y^2)^{-3/2} \end{aligned} \tag{4}$$

This system of two coupled second-order conservative differential equations can be recast into a system of the four coupled first-order equations

$$\begin{aligned} \frac{dx}{dt} &= a \\ \frac{da}{dt} &= -2x(1 + 2x^2 + 2y^2)^{-3/2} \\ \frac{dy}{dt} &= b \\ \frac{db}{dt} &= -2y(1 + 2x^2 + 2y^2)^{-3/2} \end{aligned} \tag{5}$$

This equation was solved using both the fourth-order Runge-Kutta method and the Leepfrog method. The initial conditions were taken to be $x(0) = 1$, $\dot{x}(0) = 0$, $y(0) = 0$, and $\dot{y}(0) = 0.1$, and the solutions were integrated out to $t = 100$. Step sizes of 1, 0.5, 0.25, and 0.1 were used. The solutions using the Runge-Kutta method are shown in figure 5, and the solutions from the Leapfrog method are plotted in figure 6. Neither method achieves good convergence until $h = 0.25$.

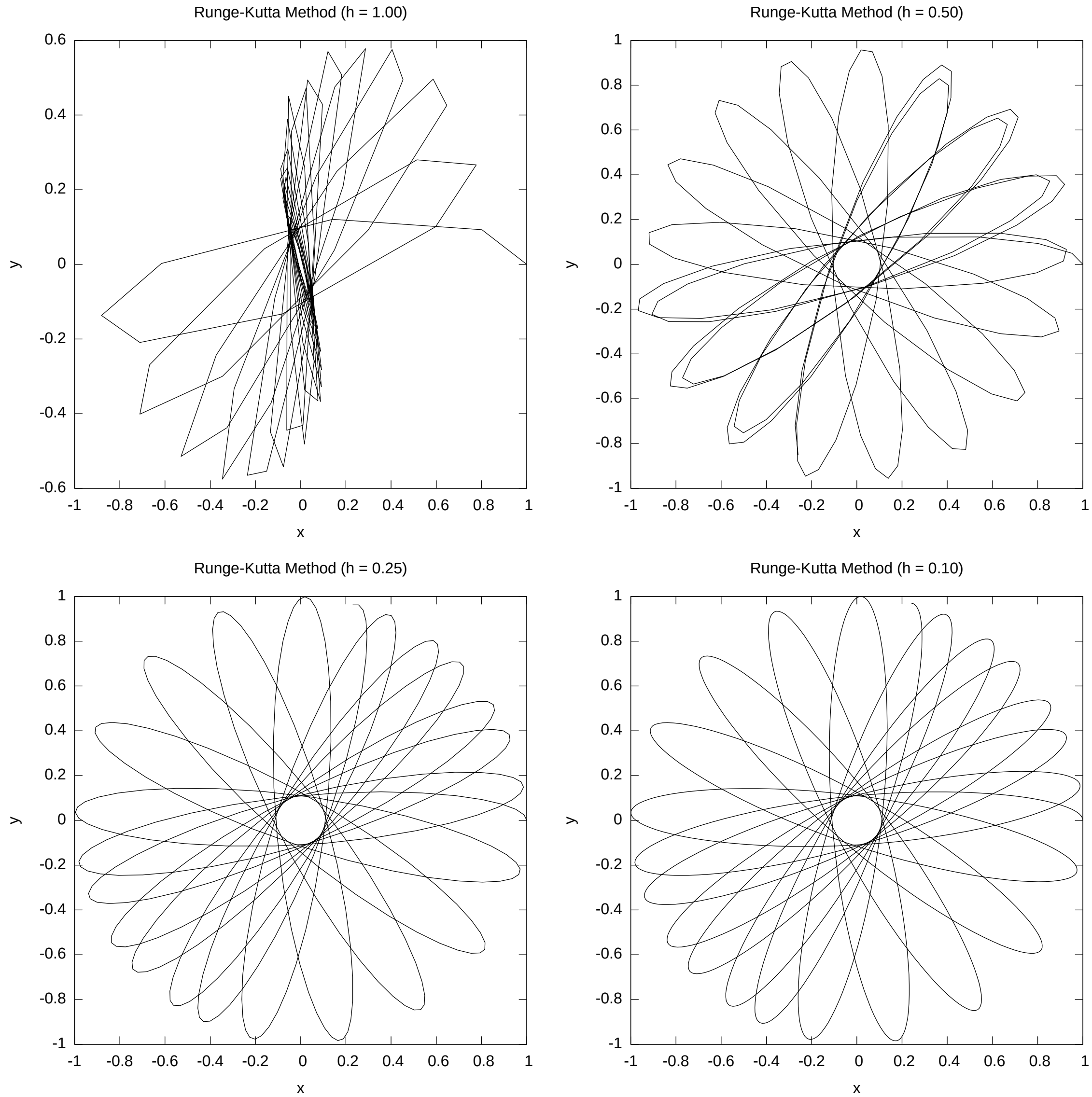

FIG. 5: Orbits from the Runge-Kutta solution of equation 5 for various time steps h.

The energy per unit mass of the particle is given by

$$\varepsilon = \frac{\dot{x}^2 + \dot{y}^2}{2} + \Phi(x, y) \tag{6}$$

The velocities needed to calculate the energy are computed as a and b in the Runge-Kutta solution of equation 5, but must be extracted from the Leapfrog method by resyncing the offset velocities using half Euler steps. The energy of the orbit is plotted against t for both solvers in figure 7.

The energy for $h = 1$ is meaningless for both solvers, as neither produced stable orbits for this time step. However, for all smaller values of h, the energy of the Leapfrog solution is steady over long periods of time while the energy of the Runge-Kutta solution steadily decreases with time. Thus, while the local variations in the energy of the Leapfrog solution may be much larger than those of the Runge-Kutta solution, the energy is more stable over long periods of time, indicating that the Leapfrog method has automatically enforced energy conservation for this system.

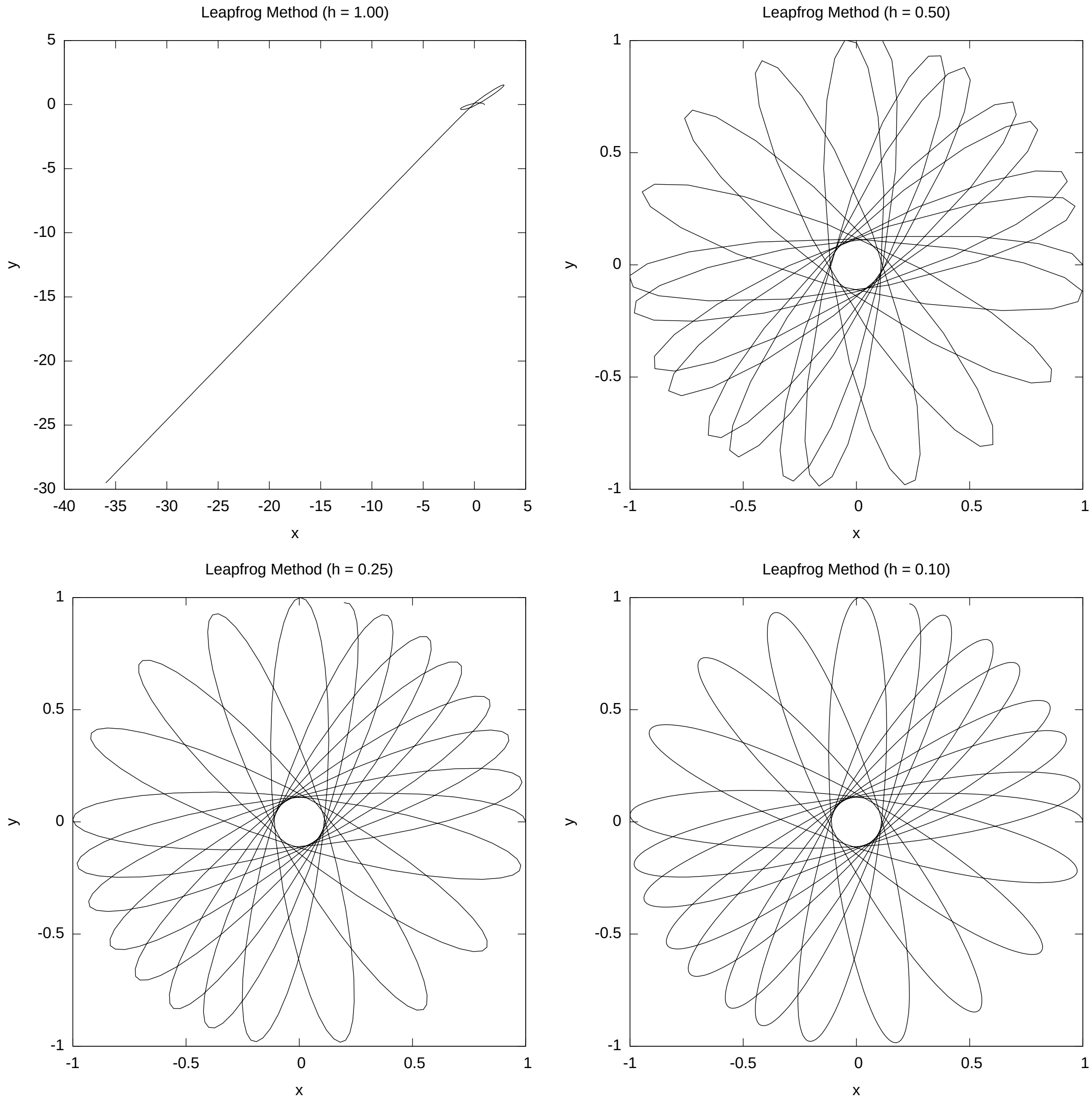

FIG. 6: Orbits from the Leapfrog solution of equation 4 for various time steps h.

III. THE LOTKA-VOLTERRA PREDATOR-PREY MODEL

The populations of two species in a predator-prey relationship, such as rabbits and foxes, can be described by the first-order Lotka-Volterra equations

$$\begin{aligned} \dot{x} &= (A-d)x - Bxy \\ \dot{y} &= (-C-e)y + Dxy \end{aligned} \tag{7}$$

where x and y are the population densities of the prey and predator respectively. A is the prey's reproduction rate, B is the prey's consumption rate by the predator, C is the predator's death rate, D is the predator's population growth rate due to consumption of the prey, and d and e are the hunting rates of the prey and predator respectively.

Here we consider an ecosystem containing rabbits and foxes with parameters $A = 1$, $B = 0.1$, $C = 1.5$, $D = 0.03$, and $d = e = 0$. The initial conditions are taken to be $x(0) = 30$ and $y(0) = 3$. The fourth-order Runge-Kutta integrator was used to integrate the solution out to $t = 100$ for time steps 1, 0.5, 0.25, and 0.1. Phase diagrams for each time step are shown in figure 8.

Supposing that both species are hunted at an equal rate q (so $d = e = q$), there should be some value of q such that both populations become extinct by $t = 100$ (where extinction is defined by $x, y < 10^{-9}$). To find this value of q, a

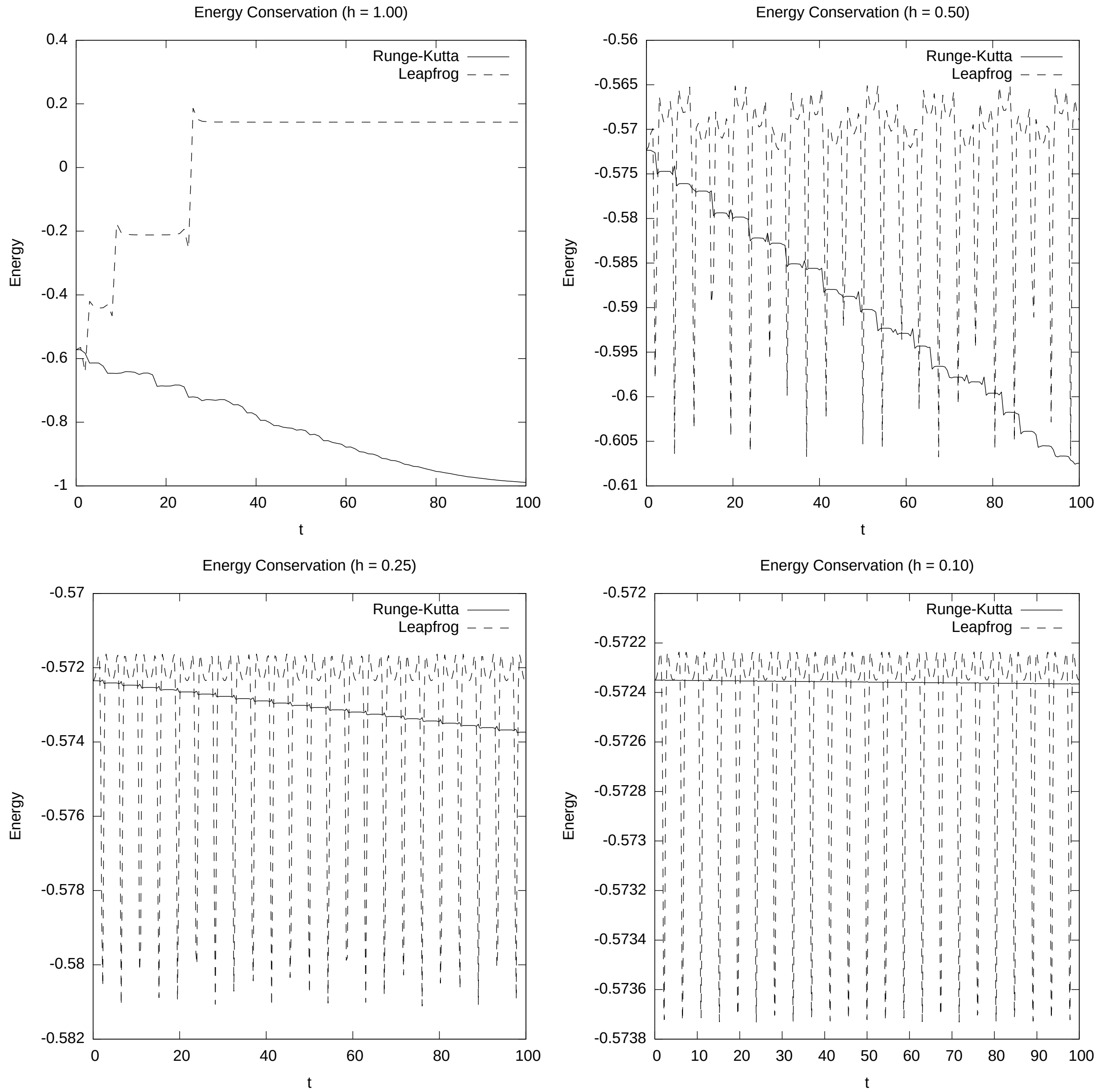

FIG. 7: Energy Conservation.

bisection-style method was used. The value was first bracketed by observing that for $q = 0$, $x(100), y(100) > 10^{-9}$, and for $q = 5$, $x(100), y(100) < 10^{-9}$. The system was then solved choosing q at the midpoint of this interval. If both species were found to be extinct at $t = 100$, the upper bound was set equal to the midpoint. Otherwise the lower bound was set equal to the midpoint. This procedure was repeated until the size of the interval fell below 10^{-4}, obtaining 4 digits of accuracy. Using a time step of $h = 0.1$, it was found that $q = 1.2398$ is the minimum hunting rate necessary to drive both species to extinction by $t = 100$.

[1] Press, William H., Saul A. Teukolsky, William T. Vetterling, Brian P. Flannery. *Numerical Recipes in FORTRAN 77: The Art of Scientific Computing (Volume 1 of Fortran Numerical Recipes)*. Second Edition. Cambridge University Press, 2001.

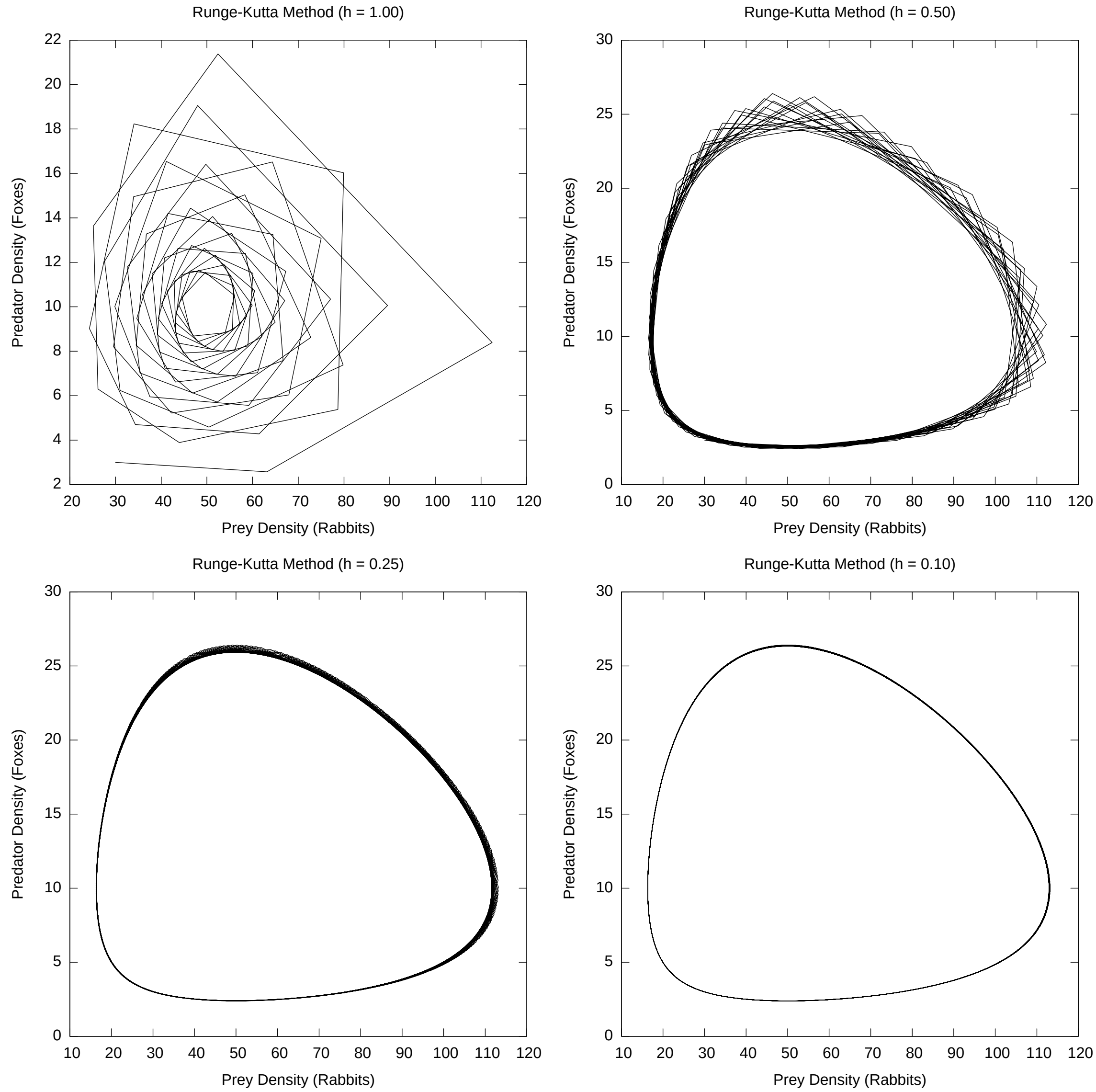

FIG. 8: Phase diagrams for the Lotka-Volterra Predator-Prey model (equation 7) integrated using the Runge-Kutta method with various time steps.