ASTR415: Problem Set #4

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The role of random numbers in scientific computing was explored through their use in data generation and Monte Carlo integration. First, a function was written that successfully transforms a uniformly distributed deviate into a Rayleigh distributed deviate. Second, a density function was integrated over a volume to determine the mass contained in that volume. The mass was found to be $M = 529.52 \pm 0.23$, and the estimated error decreased with $\sqrt{N}$ as expected.

I. RAYLEIGH DISTRIBUTED DEVIATES

A Rayleigh distributed deviate is given by

$$p(y)dy = ye^{-y^2/2}dy$$

Thus, if x is a uniform deviate,

$$\frac{dx}{dy} = ye^{-y^2/2}$$

Integrating and solving for y yields

$$y = \sqrt{-2\ln(x)}$$

The program `rayleighdeviates` uses a random number generator (such as `ran3` from [1]) to create 1024 uniform deviates, which are then transformed to Rayleigh distributed deviates and written to standard out. A normalized histogram plot of these deviates, along with a graph of $p(y) = ye^{-y^2/2}$, is shown in figure 1.

The histogram and the probability distribution function match up closely, indicating that the transformation was done correctly.

II. MONTE CARLO INTEGRATION

Given a density function ρ, the mass contained in a volume V of space can be calculated as

$$M = \int_V \rho d\tau$$

If N sample points are used to estimate the average value of the density in the volume, then the Mean Value Theorem allows the integral to be estimated using

$$M = \int_V \rho d\tau \approx V\langle\rho\rangle \pm \sqrt{\frac{\langle\rho^2\rangle - \langle\rho\rangle^2}{N}}$$

Here, the density function is given by

$$\rho(x, y, z) = 1 + x^2 + 3(y + z)^2$$

and V is defined by $x^2 + y^2 + z^2 \leq 9$, $x \geq 0$, $y \geq -1$. Geometrically, this is a sphere of radius 3 sliced along the planes $x = 0$ and $y = -1$. Thus, it is bounded by a rectangular prism of dimensions $3 \times 4 \times 6$ centered at the point $(\frac{3}{2}, 2, 0)$. The sampling is done by choosing random points inside this prism and rejecting those whose distance from the origin is greater than 3.

Monte Carlo integration was performed on this volume for N ranging from 10^1 to 10^7 in integer powers of 10. As N increased, the estimated error decreased by approximately 3 for each power of 10. This confirms that the error is reduced only as $\sqrt{N}$, since $\sqrt{10} \approx 3$. A plot of the estimated integral and error versus $\log(N)$ is shown in figure 2. Using 10^7 sample points, the final estimate for the mass of the region is $M = 529.52 \pm 0.23$.

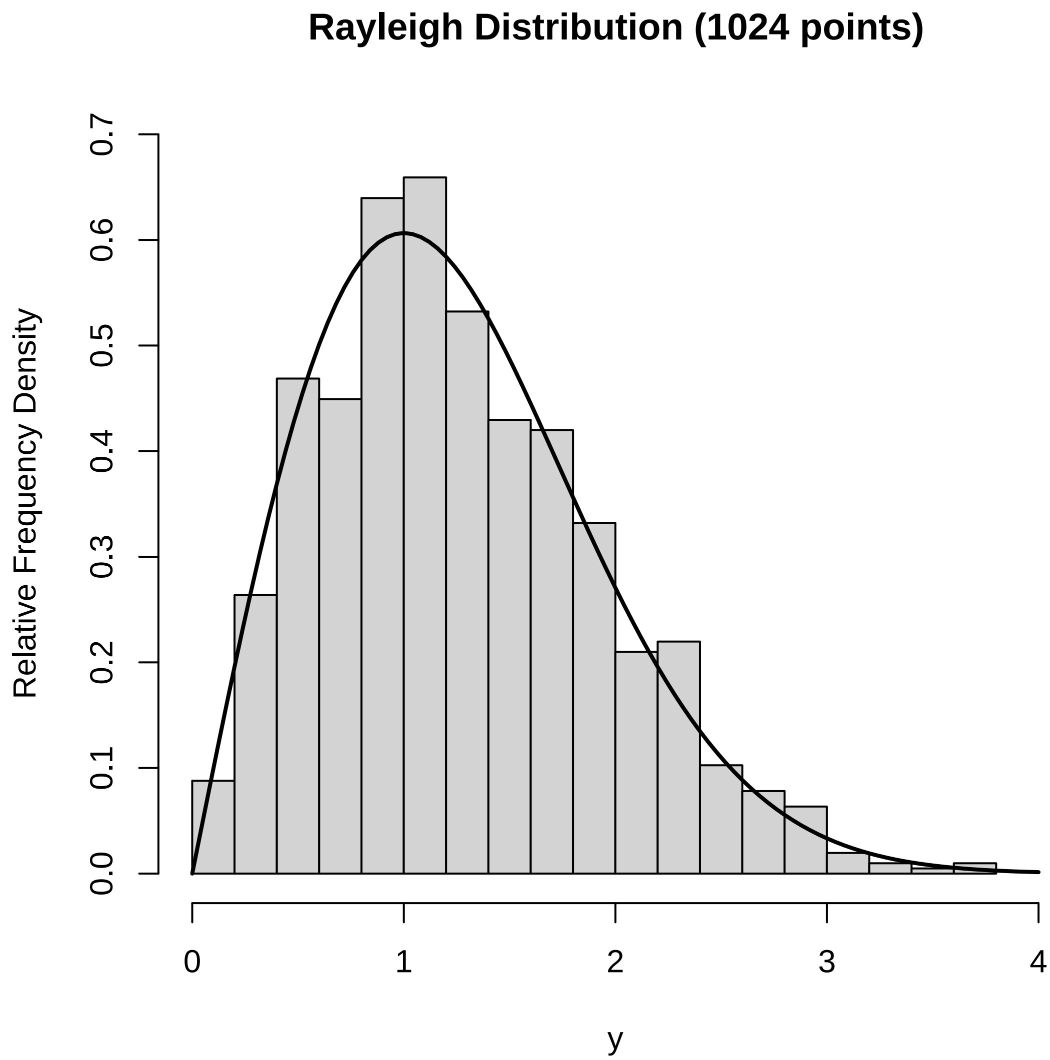

FIG. 1: Normalized histogram for 1024 Rayleigh distributed deviates overlayed with a plot of the theoretical probability distribution function.

[1] Press, William H., Saul A. Teukolsky, William T. Vetterling, Brian P. Flannery. *Numerical Recipes in FORTRAN 77: The Art of Scientific Computing (Volume 1 of Fortran Numerical Recipes)*. Second Edition. Cambridge University Press, 2001.

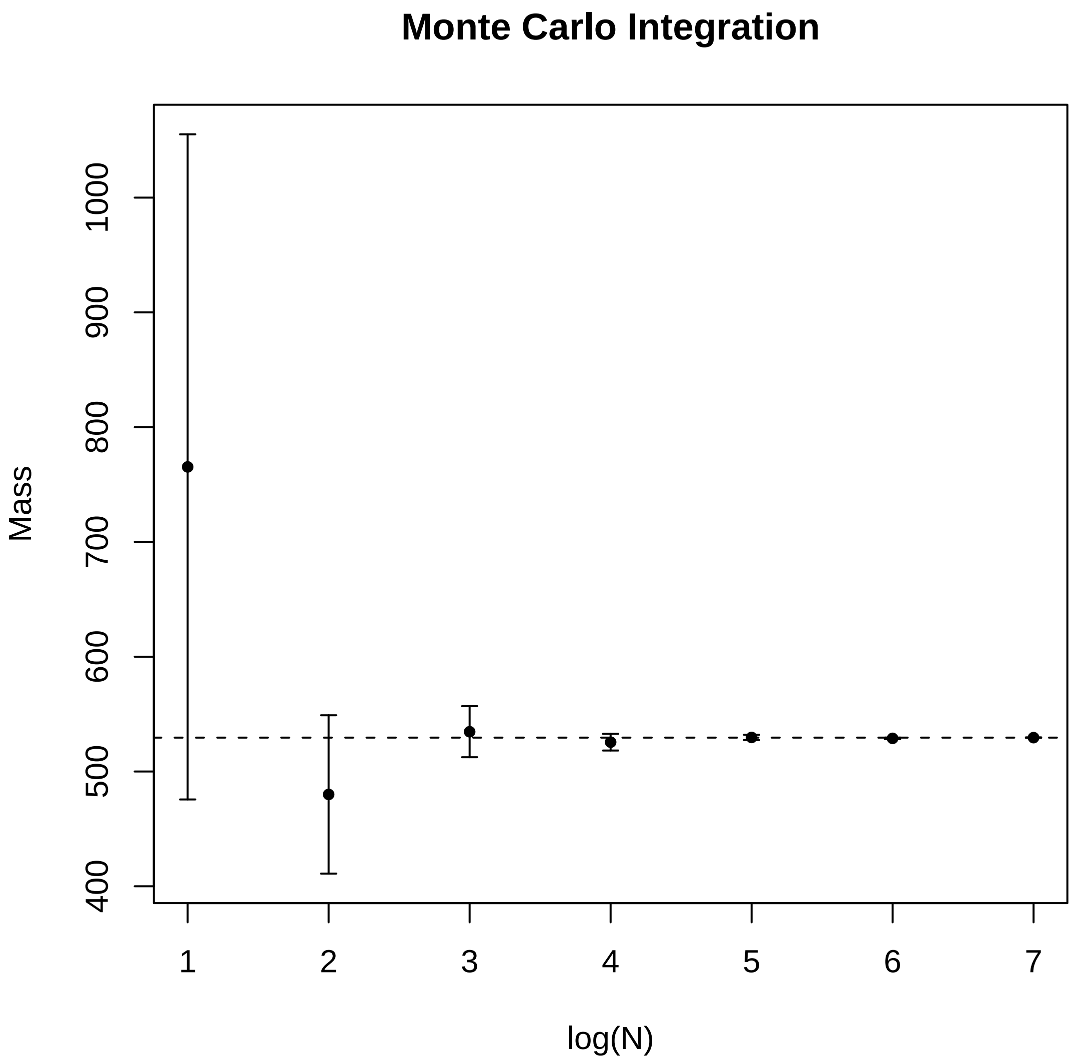

FIG. 2: Monte Carlo estimate of the mass contained in V with estimated error plotted against the number of sample points.